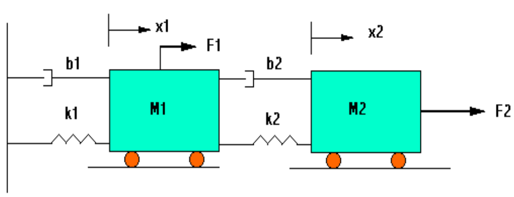
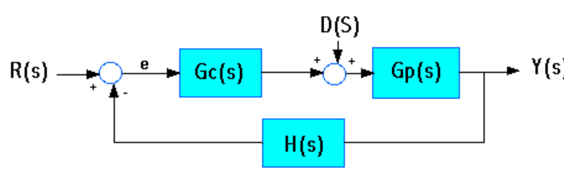
CONTINOUS

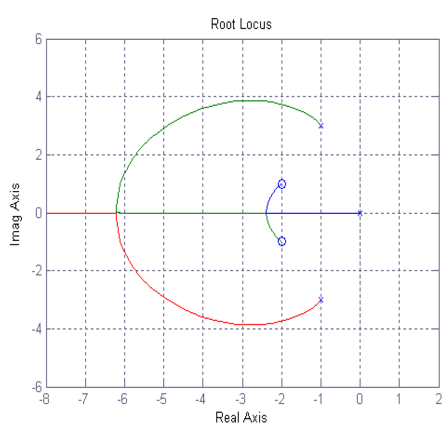
CONTROL OF

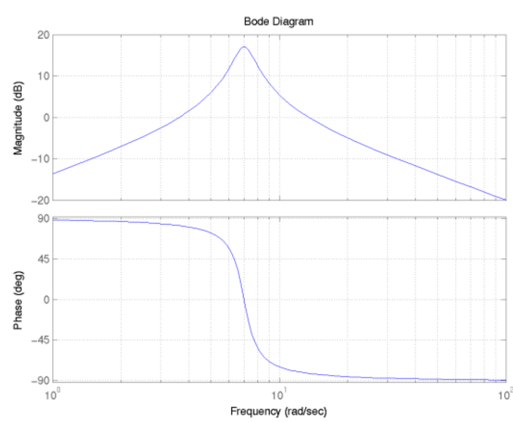
ENGINEERING SYSTEMS

First Edition









DAVID S. DEWOLF

Brief Contents

Chapter

1

Ordinary Differential Equations

Solution of Linear, Time-Invariant, O.D.E.'s

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots\dots\dots a_1 \frac{dx}{dt} + a_0 x = f(t)$$

Solution: $x(t) = x_H(t) + x_p(t)$

Where $x_H(t)$ is the "Homogeneous" solution, set RHS = 0 , system not affected by f(t)

Where $x_P(t)$ is the "Particular" solution, system affected by f(t), "outside influences"

In General:

$x_H(t)$ = Ce^{st} where C, s are constants

$x_P(t)$ = $B_o f(t) + B_1 \frac{df}{dt} + \dots\dots\dots\dots B_n \frac{d^n f}{dt^n}$ where B_o, B_1,…B_n are constants

Homogeneous:

$x_H(t)$ = Ce^{st}

$\dot{x}_H(t)$ = sCe^{st}

$\ddot{x}_H(t)$ = s^2Ce^{st}

etc…..

$$a_n s^n C e^{st} + a_{n-1} s^{n-1} C e^{st} + \dots\dots\dots\dots a_1 s C e^{st} + a_0 C e^{st} = 0$$

$[\, a_n s^n + a_{n-1} s^{n-1} + \dots\dots\dots\dots a_1 s + a_0 \,]$ Ce^{st} = 0

either Ce^{st} = 0

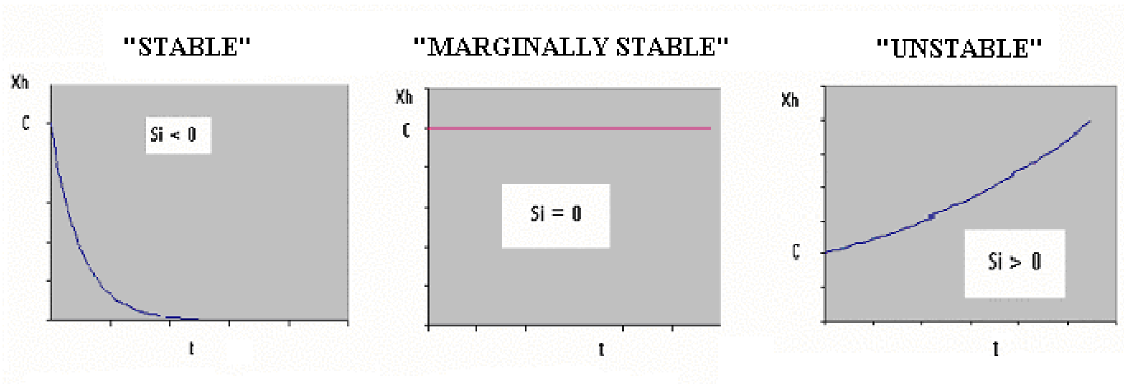
or

$a_n s^n + a_{n-1} s^{n-1} + \dots\dots\dots\dots a_1 s + a_0 = 0$ This is "Characteristic Equation"

There are n roots: S_1, S_2, ……., S_n which satisfy this equation

For all a_i's Real, the roots are either real or complex conjugate pairs

(i) Real Pair $s = s_i$ $\therefore$ $x_H(t) = Ce^{s_i t}$



In the usual case where x represents some variable in a system-displacement, voltage, current, pressure, flow rate, temperature, etc… System must be stable or it will destroy itself!!

(ii) Complex Conjugate Pairs $s_{i,i+1} = a \pm ib$

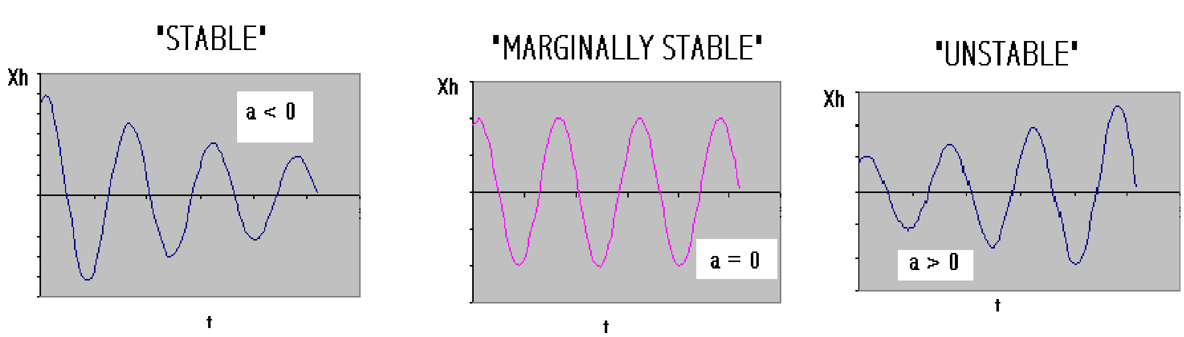
Take in the form of Euler's Identity:

$x_H(t) = Ce^{at}\sin(bt + \phi)$

Reminder from math identity that $\sin(A + B) = \sin A\cos B + \cos A\sin B$

or $x_H(t) = C_1e^{at}\sin(bt) + C_2e^{at}\cos(bt)$

where C, C_1, C_2, ϕ, are constants, b is in rad/sec



Period is T = $\dfrac{2\pi}{b}$

Particular Solution has three common cases:

(i) f(t) = constant $\therefore$ $x_P(t)$ = constant

$$\dot{x}_P(t) = 0$$

$$\ddot{x}_P(t) = 0$$

etc….

$$a_o x_p = f \text{ or } x_P \frac{f}{a_o}$$

(ii) f(t) = ramp = $f_o + f_1 t$ $\therefore$ $x_P = B_o + B_1 t \quad \dot{x}_P = B_1$

$$\dot{x}_P = B_1$$

$$\ddot{x}_P = 0$$

$$a_1 B_1 + a_o(B_o + B_1 t) = f_o + f_1 t$$

$$a_o B_o + a_1 B_1 = f_o \quad (1)$$

$$a_o B_1 = f_1 \qquad (2)$$

Two Equations and Two Unknowns

(iii) f(t) = sinusoidal = A$\sin\omega t$ $\therefore$ $x_P = A\sin\varpi t + B\cos\varpi t$

$$\dot{x}_P = \varpi A\cos\varpi t - \varpi \sin\varpi t$$

$$\ddot{x}_P = -\varpi^2 A\sin\varpi t - \varpi^2 B\cos\varpi t$$

Step Response 1st Order, Stable Systems

$x(t) = x_H(t) + x_p(t)$

$x_H(t)$ = Ce^{st} and $x_P(t)$ = constant

<u>Time Constant</u> is the time required for a stable system to decay from 0.37 of the difference between start and end values.

$$\tau = \frac{1}{-s} \quad \text{(seconds)}$$

<u>Settling Time</u> is 4 time constants (usually) or approximately 98% of the decay

$$T = \frac{4}{-s} \quad \text{(seconds)}$$

Step Response 2nd Order, Stable Systems

$\ddot{x} + a_1\dot{x} + a_o x = 1$ where a_1 and a_o are constants

$x(t) = x_H(t) + x_p(t)$

$x_H(t):\ s^2Ce^{st} + a_1 sCe^{st} + a_o Ce^{st} = 0$

$\therefore$ $s^2 + a_1 s + a_o = 0$ is the "Characteristic Equation"

Roots are $s_{1,2} = \dfrac{-a_1 \pm \sqrt{a_1^{\,2} - 4a_o}}{2}$ Stable <u>iff</u> $a_1 > 0$, $a_o > 0$

Now define some new variables

$\omega_n^{\,2} = a_o$, ω is the "Natural Frequency"

$2\xi\,\omega_n = a_1$, ξ is the "Damping Ratio"

Solving the "Characteristic Equation" is now $s^2 + 2\xi\,\omega_n s + \omega_n^{\,2} = 0$

Roots are $s_{1,2} = \dfrac{-2\xi\varpi_n \pm \sqrt{4\xi^2\varpi_n^{\,2} - 4\varpi_n^{\,2}}}{2}$

$$s_{1,2} = -\xi\varpi_n \pm \varpi_n\sqrt{\xi^2 - 1}$$

Use fullness of the "Damping Ratio"

(i) $\xi > 1$ $\therefore$ two real roots $s_{1,2}$

$x_H(t) = C_1 e^{s1t} + C_2 e^{s2t}$ such a system is called "Over Damped"

$$s_{1,2} = -\xi\varpi_n \pm \varpi_n\sqrt{\xi^2 - 1}$$

$$\tau = \frac{1}{-\xi\varpi_n + \varpi_n\sqrt{\xi^2 - 1}}$$

(ii) $\xi < 1$ $\therefore$ roots are complex conjugate

$x_H(t) = Ce^{-\xi\omega n\, t}\sin(\omega_d t + \phi)$ such a system is "Under Damped"

$$s_{1,2} = -\xi\varpi_n \pm i\,\varpi_n\sqrt{1-\xi^2}$$

$$\tau = \frac{1}{\xi\varpi_n}$$

Define: $\omega_d = \varpi_n\sqrt{1-\xi^2}$ which is called "Damped Natural Frquency"

Period = T = $\dfrac{2\pi}{\varpi_d}$

(iii) $\xi = 1$ $\therefore$ roots are real and repeated

$$x_H(t) = [C1 + C2t]e^{-\xi\omega nt}$$

$s_{1,2} = -\xi\varpi_n$ such a system is "Critically Damped"

$$\tau = \frac{1}{\xi\varpi_n}$$

(iv) $\xi = 0$ $\therefore$ roots are a complex pair with real root = 0

$x_H(t)$ = Csin($\omega_n t$ + ϕ) such a system is "Undamped"

$$s_{1,2} = \pm i\varpi_n$$

Period = T = $\dfrac{2\pi}{\varpi_n}$

Summary:

If $\xi \geq 1$, Behavior is <u>purely exponential</u>, both roots are real

If $\xi < 1$, Behavior is <u>sinusoidal</u> with exponential decaying amplitude and frequency ω_d

At limit $\xi = 0$, Behavior is sinusoidal with constant amplitude and frequency ω_n

In all cases, the particular solution is $x_p = \dfrac{1}{\varpi_n^{\ 2}}$

All roots are either real or complex conjugate pairs

(1st Order) Real s = -a $\therefore$ $x_H(t)$ = Ce^{-at} and $\tau = \dfrac{1}{-a}$

(2nd Order) Complex Conjugate s = -a ± *i*b $\therefore$ $x_H(t)$ = Ce^{-at}sin(b*t* + ϕ)

$$\tau = \frac{1}{-a}\ ,\ T = \frac{2\pi}{b}\ ,\ \xi\,\omega_n = a\ ,\ \omega_d = b$$

"Dominant Root" in a stable system: that is the root(s) with the "Least Negative" real part

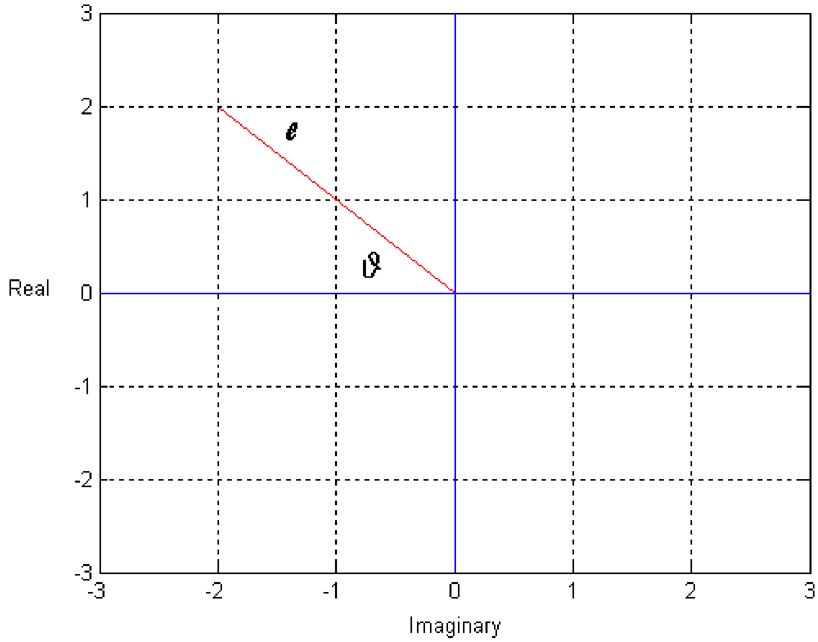
$$x_H(t) = C_1e^{s1t} + C_2e^{s2t} + ... + C_ne^{snt}$$

each root has its own τ

each root "settles" at its own 4τ

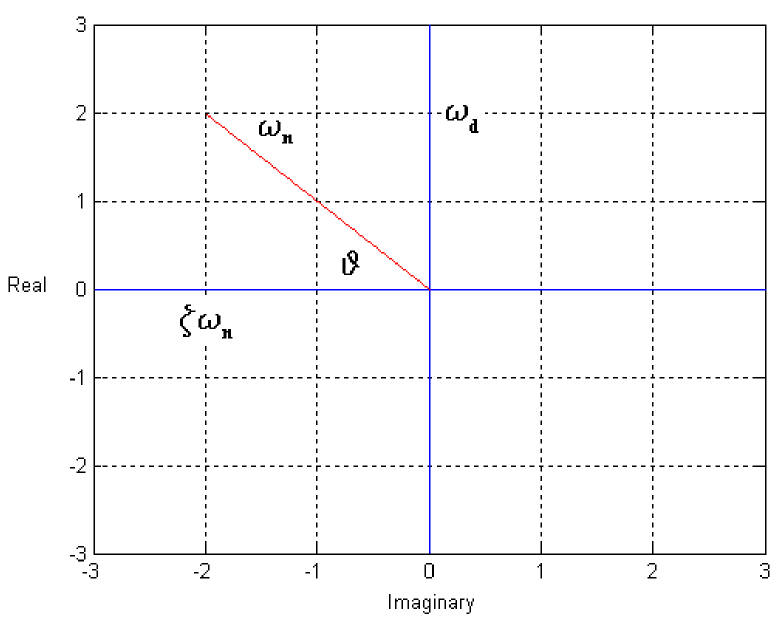
Slowest root to settle is Dominant Root

There are also some Graphical representations for ξ , ω_n , ω_d



$$\ell = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} \qquad \ell = \sqrt{\xi^2\varpi_n{}^2 + \omega_n{}^2(1-\xi^2)}$$

$$\ell = \omega_n \qquad \cos\theta = \frac{\xi\varpi_n}{\varpi_n} = \xi$$



EXAMPLES

Example # 1 Given $\ddot{x} + 4\dot{x} + 29x = 5 + 2t$ with $x(0) = 0$ and $\dot{x}(0) = 1$ (Ramp)

Characteristic Equation is $s^2 + 4s + 29 = 0$

$S_{1,2} = 2 \pm 5i$

$x_H(t) = Ce^{-2t}\sin(5t + \phi)$ or $x_H(t) = C_1e^{-2t}\sin 5t + C_2e^{-2t}\cos 5t$

Ramp Fn

$x_P(t) = at + b$

$$\frac{d^2xp}{dt^2} + 4\frac{dxp}{dt} + 29dxp = 5 + 2t$$

$$0 + 4a + 29(at + b) = 5 + 2t$$

$$4a + 29at + 29b = 5 + 2t$$

Only True if

(1) $29at = 2t$

(2) $4a + 29b = 5$

Solving the two equations

$$a = \frac{2}{29}$$

$$b = \frac{137}{841}$$

$$x_P(t) = \frac{2}{29}t + \frac{137}{841}$$

$$\therefore \quad x(t) = C_1 e^{-2t}\sin 5t + C_2 e^{-2t}\cos 5t + \frac{2}{29}t + \frac{137}{841}$$

$$\dot{x}(t) = C_1(-2e^{-2t}\sin 5t + 5e^{-2t}\cos 5t) + C_2(-2e^{-2t}\cos 5t - 5e^{-2t}\sin 5t) + \frac{2}{29}$$

Apply initial conditions:

(1) $x(0) = 0 = 0 + C_2 + \frac{137}{841}$

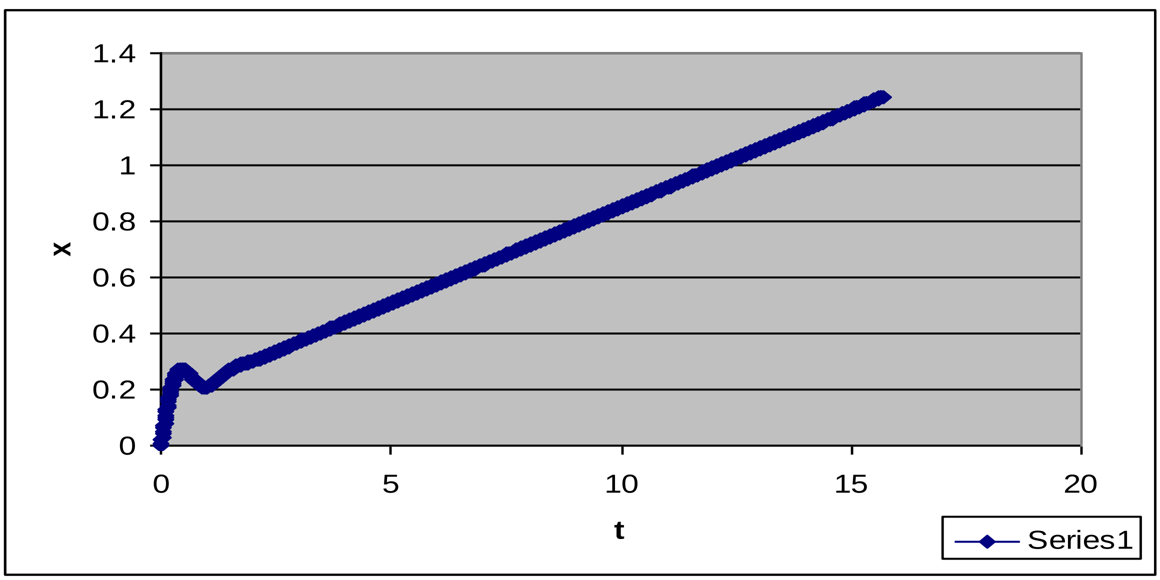
(2) $\dot{x}(0) = 1 = C_1(0 + 5) + C_2(-2 + 0) + \frac{2}{29}$

Solving the two equations

$C_1 = \frac{509}{4205}$

$C_2 = \frac{-137}{841}$

$$\therefore \quad x(t) = \frac{509}{4205}e^{-2t}\sin 5t - \frac{137}{841}e^{-2t}\cos 5t + \frac{2}{29}t + \frac{137}{841}$$



Example # 2 Given $\ddot{x}+3\dot{x}+2x=4$ with $x(0)=5$ and $\dot{x}(0)=0$ (Constant)

Characteristic Equation is $s^2 + 3s + 2 = 0$

$S_1 = -1$

$S_2 = -2$

$x_H(t) = C_1e^{-t} + C_2e^{-2t}$

Constant Fn

xp = k

$$\frac{d^2xp}{dt^2}+3\frac{dxp}{dt}+2dxp=4$$

$0+3*0+2k=4$

k = 2

xp = 2

$\therefore\ x(t) = C_1e^{-t} + C_2e^{-2t} + 2$

$\dot{x}(t)=-C_1e^{-t}+-2C_2e^{-2t}+2$

Apply Initial Conditions

(1) $x(0) = 5 = C_1 + C_2 + 2$

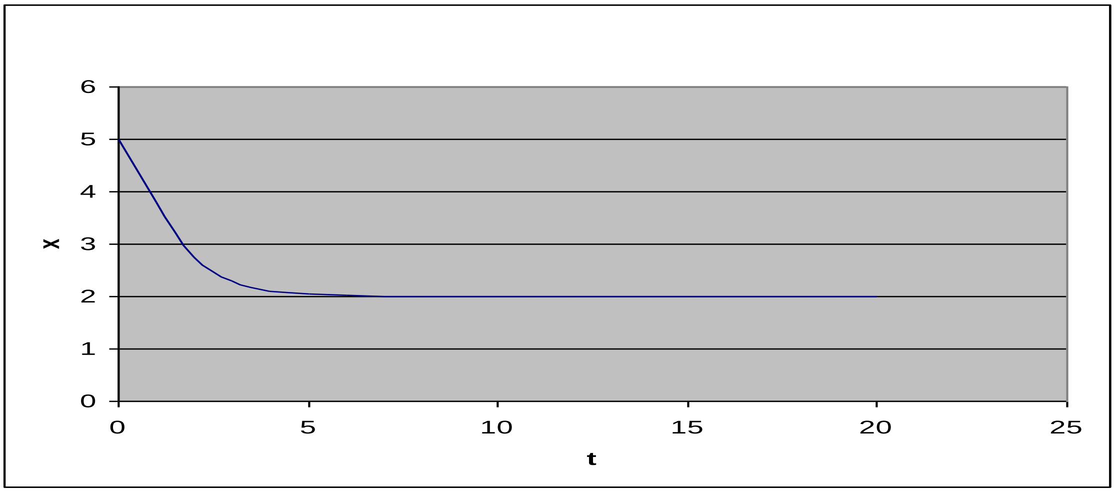
(2) $\dot{x}(0)=0 = -C_1 - 2C_2$

Solving the two equations

$C_1 = 6$

$C_2 = -3$

$\therefore\ x(t) = 6e^{-t} - 3e^{-2t} + 2$



Example # 3 Given $\ddot{x} + 6\dot{x} + 8x = 3\sin(3t)$ with $x(0) = 0$ and $\dot{x}(0) = 0$ (Sinusoid)

Characteristic Equation is $s^2 + 6s + 8 = 0$

$S_1 = -4$

$S_2 = -2$

$x_H(t) = C_1e^{-4t} + C_2e^{-2t}$

Sinusoid Fn

xp = Asin3t + Bcos3t

$\dot{x}p = 3A\cos 3t - 3B\sin 3t$

$\ddot{x}p = -9A\sin 3t - 9B\cos 3t$

$\therefore$ -9Asin3t - 9Bcos3t + 6[3Acos3t - 3Bsin3t] + 8[Asin3t + Bcos3t] = 3sin3t

Collect Terms

(1) sin3t(-9A - 18B + 8A) = 3sin3t

(2) cos3t(-9B + 18A + 8B) = 0

(1) –A +18B = 3

(2) –B +18A = 0

Two Equations and two unknowns

A = $\frac{-3}{325}$

B = $\frac{-54}{325}$

xp = $-\frac{3}{325}\sin 3t - \frac{54}{325}\cos 3t$

$\therefore\ x(t) = C_1e^{-4t} + C_2e^{-2t} - \frac{3}{325}\sin 3t - \frac{54}{325}\cos 3t$

$\dot{x}(t) = -4C_1e^{-4t} - 2C_2e^{-2t} - \frac{9}{325}\cos 3t - \frac{162}{325}\sin 3t$

Apply Initial Conditions

(1) $x(0) = 0 = C_1 + C_2 - \frac{54}{325}$

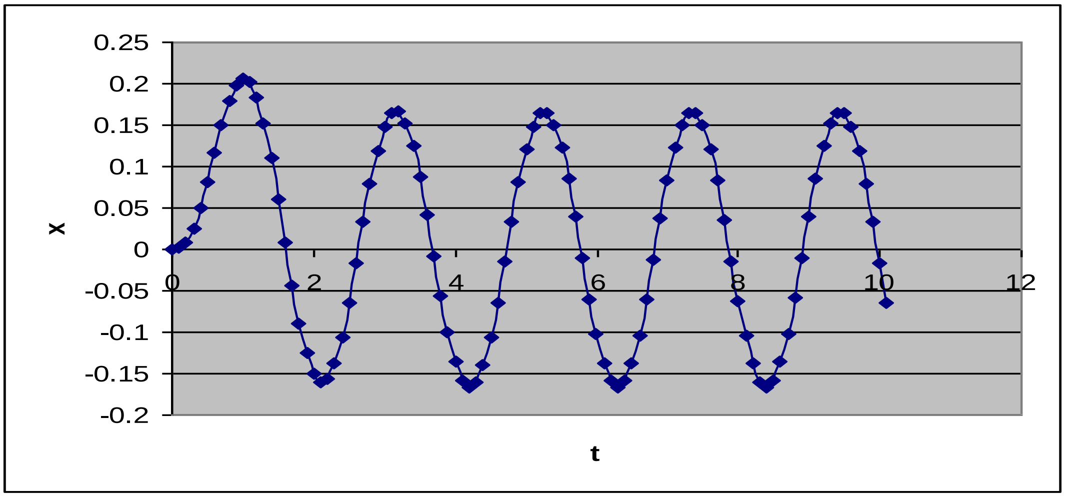
(2) $\dot{x}(0) = 0 = -4C_1 - 2C_2 - \frac{9}{325}$

Two equations and two unknowns

$C_1 = \frac{-117}{650}$

$C_2 = \frac{225}{650}$

$\therefore\ x(t) = -\frac{117}{650}e^{-4t} + \frac{225}{650}e^{-2t} - \frac{3}{325}\sin 3t - \frac{54}{325}\cos 3t$



Example # 4 Special Case: Given $\ddot{x}+0.4\dot{x}+4x=f(t)$

Characteristic Equation is $s^2 + 0.4s + 4 = 0$

$S_{1,2} = -0.2 \pm 1.99i$

$x_H(t) = Ce^{-.2t}\sin(1.99t+\phi)$ or $x_H(t) = C_1e^{-.2t}\sin 1.99t + C_2e^{-.2t}\cos 1.99t$

$\tau = \dfrac{1}{.2} = 5\,\text{seconds}$

Settling Time is $T = \dfrac{4}{.2}$ = 20 seconds

$\omega_n^{\,2} = \sqrt{4}$, $\omega_n = 2$

$2\xi\omega_n = 0.4$, $\xi = 0.04$, $\xi < 1$, $\omega_d = \varpi_n\sqrt{1-\xi^2} = 2rad/\sec$, Period = $\dfrac{2\pi}{\varpi_d}$ = 3.14 sec

Find the particular solution if $f(t) = \begin{cases} 0, 1 > t > 0 \\ 1, t \geq 1 \end{cases}$

Case 1 (Constant) 4k = 0, k = xp = 0

Case 2 (Constant) 4k = 1, k = xp = $\dfrac{1}{4}$

Now apply initial conditions only to homogeneous part!!!!

(1) $x(0) = 1 = C_2$

(2) $\dot{x}(0) = 0 = C_1(0 + 1.99) + C_2(-0.2 + 0)$

Two equations and two unknowns

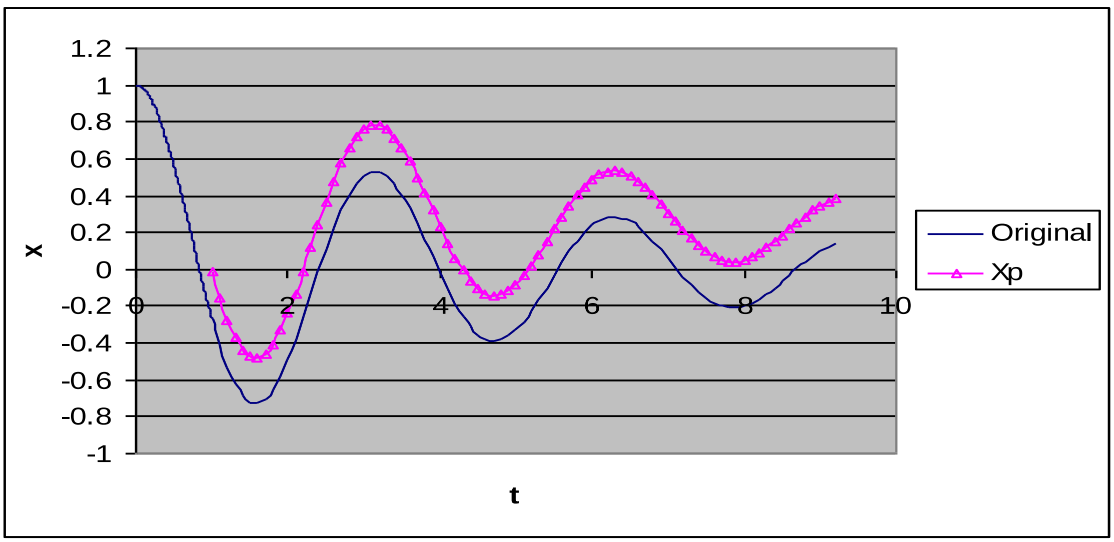
$C_1 = 0.1$

$C_2 = 1.0$

$\therefore$ For the first part the solution is

$x(t) = 0.1e^{-0.2t}\sin 1.99t + e^{-0.2t}\cos 1.99t$ up until time reaches @ t =1

Then the particular solution takes in affect after that xp + xh



General Type of Questions:

(1) Suppose the eigenvalues of the A-Matrix are

$s_{1,2} = -0.5 \pm i\pi$ and $s_{3,4} = -0.1 \pm i$

Dominant roots are $s_{3,4} = -0.1 \pm i$, “Least Negative”

What’s size of A-matrix? 4 x 4

What’s Settling Time? T = $\frac{4}{.1} = 40$ seconds

Write General Solution: $x_H(t) = C_1 e^{-0.5t}\sin(\pi t + \phi_1) + C_2 e^{-0.1t}\sin(1t + \phi_2)$

(2) Assume a model by $2\ddot{x} + 2\dot{x} + 50x = f(t)$

Rewrite as $s^2 + s + 25 = 0$

$s_{1,2} = -0.5 \pm 4.97i$

What’s Natural frequency? $\omega_n^2 = \sqrt{25}$, $\omega_n = 5$ rad/sec

What’s the damping ratio? $2\xi\omega_n = 1$, $\xi = 0.1$

What’s the time constant? $\tau = \frac{1}{.5}$ = 2 seconds

What’s the Period?

Since $\xi < 1$, $\omega_d = \varpi_n\sqrt{1-\xi^2} = 4.97 rad/\text{sec}$ $\therefore$ Period = $\frac{2\pi}{\varpi_d}$ = 1.26 sec

(3) Assume the system modeled by $\ddot{x} + 3\dot{x} + 10x = 4\sin 3t$

$s_{1,2} = -1.5 \pm 2.78i$, The system is STABLE $a < 0$

(4) Assume the system modeled by $\ddot{x} + 18\dot{x} = 0$

$s_1 = 0$ and $s_2 = -18$, The system is MARGINALLY STABLE $a = 0$

(5) Assume the system modeled by $\ddot{x} - 2\dot{x} - 12x = e^{-2t}$

$s_1 = 4.6$ and $s_2 = -2.6$, The system is UNSTABLE $a > 0$

Chapter

2

CONTINUOUS CONTROL

MAE 443

State Space

State Space

"State Space" is the mathematical domain comprised of the "states" of a system

"State" numerous (infinite!) definitions possible for any linear system

Typically: Displacement, Velocity, Current, Voltage, Pressure, Flow rate, Temperature, etc….

Form of state space equations:

Let: q_1 ,q_2, ……..q_n = states (n^{th} Order system)

State – Space dynamic equations consist of the n, 1^{st} order, O.D.E.'s of the states

$$\dot{q}_1 = \ldots\ldots\ldots\ldots$$

$$\dot{q}_2 = \ldots\ldots\ldots\ldots$$

.
.
.
.

$$\dot{q}_n = \ldots\ldots\ldots\ldots$$

If the system is <u>linear</u>, then the RHS's are made up of two kinds of terms:

Linear combination of states:

$$\dot{q}_1 = a_{11}q_1 + a_{12}q_2 + \ldots.a_{1n}q_n$$
$$\dot{q}_2 = a_{21}q_1 + a_{22}q_2 + \ldots a_{2n}q_n$$

.
.
.

Linear combinations of inputs:

$$\dot{q}_1 = \ldots b_{11}u_1 + b_{12}u_2 + \ldots b_{1m}u_m$$
$$\dot{q}_2 = \ldots b_{21}u_1 + b_{22}u_2 + \ldots b_{2m}u_m$$

.

.

.

"Pay very close attention to sub-scripts !!!!"

"SIZE MATTERS"

Therefore State space takes on this form:

$$\dot{q}_1 = a_{11}q_1 + a_{12}q_2 + \ldots a_{1n}q_n \;+ b_{11}u_1 + b_{12}u_2 + \ldots b_{1m}u_m$$
$$\dot{q}_2 = a_{21}q_1 + a_{22}q_2 + \ldots a_{2n}q_n \;+ b_{21}u_1 + b_{22}u_2 + \ldots b_{2m}u_m$$

. .

. .

. .

$$\dot{q}_n = a_{n1}q_1 + a_{n2}q_2 + \ldots a_{nn}q_n \;+ b_{n1}u_1 + b_{n2}u_2 + \ldots b_{nm}u_m$$

Therefore:

$\dot{q} =$ A$\underline{q}$ + B$\underline{u}$ is the "State Matrix"

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix} \quad , \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & a_{22} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nm} \end{bmatrix}$$

Output Equation:

In addition to the state equation, a state-space formulation normally includes an <u>output</u> equation.

$\underline{y} = C\underline{q} + D\underline{u}$

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_p \end{bmatrix} = \text{Vector of “outputs” (number of outputs = p)}$$

<u>Summary of State-Space Formulation</u>

1) $\underline{q}$ = “State Vector” size is n x 1

2) $\underline{u}$ = “Input Vector” size is m x 1

3) $\underline{y}$ = “Output Vector” size is p x 1

4) A = “State Matrix” size is n x n

5) B = “Input Matrix” size is n x m

6) C = “Output Matrix” size is p x n

7) D = “Direct Transmission Matrix” size is p x m

8) Rows x Columns

EXAMPLES

Example # 1 Given $\begin{array}{l} 3\dot{q}_1 + 2q_1 - 4q_2 = 6F_1(t) \\ 2\dot{q}_2 + 3q_2 - q_1 = 4F_1(t) \end{array}$ where $F_1(t)$ is the input

Convert to State Space:

Rewrite:

$$\dot{q}_1 = -\frac{2}{3}q_1 + \frac{4}{3}q_2 + 2F_1(t)$$
$$\dot{q}_2 = -\frac{3}{2}q_2 + \frac{1}{2}q_1 + 2F_1(t)$$

$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ and $\underline{u} = [F_1(t)]$ A = $\begin{bmatrix} -\frac{2}{3} & +\frac{4}{3} \\ +\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$ B = $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\dot{q} = \begin{bmatrix} -\frac{2}{3} & +\frac{4}{3} \\ +\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \underline{q} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \underline{u}$$

Example # 2 Given $2\ddot{x} + 3\dot{x} + 5x = 2\sin t$,where the input is sin$t$ and output is $3x + \dot{x}$

Define States

$q_1 = x$

$q_2 = \dot{x}$

$\underline{q} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ $\underline{u}$ = sin$t$

State Equations

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -\frac{5}{2}q_1 - \frac{3}{2}q_2 + u$$

Collect Terms

$$\dot{q} = \begin{bmatrix} +0 & +1 \\ -\dfrac{5}{2} & -\dfrac{3}{2} \end{bmatrix} \underline{q} \qquad + \qquad [\text{illegible}]\,\underline{u}$$

Output Equation : $3x + \dot{x}$

$\underline{y} = C\underline{q} + D\underline{u}$

$$\underline{y} = \begin{bmatrix} 3 & 1 \end{bmatrix} \underline{q} \qquad + \qquad \begin{bmatrix} 0 \end{bmatrix} \underline{u}$$

Example # 3 Given
$$m_1\ddot{x}_1 = k_1(x_2 - x_1) + (\dot{x}_2 - \dot{x}_1) + F_1$$
$$m_2\ddot{x}_2 = k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) + F_2$$

F_1 and F_2 are the inputs: $\underline{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

Outputs are
$$y_1 = k_1(x_2 - x_1)$$
$$y_2 = b_2(\dot{x}_2 - \dot{x}_1)$$

Define States

$q_1 = x_1$

$q_2 = \dot{x}_1$

$q_3 = x_2$

$q_4 = \dot{x}_2$

State Equations

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -\frac{k_1}{m_1}q_1 + \frac{k_2}{m_1}q_3 - \frac{b_1}{m_1}q_2 + \frac{b_1}{m_1}q_4 + \frac{u_1}{m_1}$$
$$\dot{q}_3 = q_4$$
$$\dot{q}_4 = -\frac{k_2}{m_2}q_1 + \frac{k_2}{m_2}q_3 - \frac{b_2}{m_2}q_2 + \frac{b_2}{m_2}q_4 + \frac{u_2}{m_2}$$

$\underline{y} = \begin{bmatrix} k_1(q_3 - q_1) \\ k_2(q_4 - q_2) \end{bmatrix}$

$\dot{q} = A\underline{q} + B\underline{u}$

$$\dot{q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1} & \frac{-b_1}{m_1} & \frac{-k_1}{m_1} & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{-k_2}{m_2} & \frac{-b_2}{m_2} & \frac{-k_2}{m_2} & \frac{b_2}{m_2} \end{bmatrix} \underline{q} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \\ 0 & \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} -k_1 & 0 & k_1 & 0 \\ 0 & -b_2 & 0 & b_2 \end{bmatrix} \underline{q} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}$$

Example # 4 Given
$$\ddot{x}_1 = 3(\dot{x}_2 - \dot{x}_1) + 11(x_2 - x_1) + 2F_1(t) + 5F_2(t)$$
$$\ddot{x}_2 = -4(\dot{x}_2 - \dot{x}_1) - 25(x_2 - x_1) + 5F_2(t) + F_1(t)$$

F_1 and F_2 are the inputs and output is
$$y_1 = 3x_1 + 5\dot{x}_2$$
$$y_2 = 4\dot{x}_1 + 4F_2(t)$$

Define States, written in different style than previous problem

$\underline{q} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ and $\underline{u} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$

State Equations

$$\dot{q}_1 = \dot{x}_1$$
$$\dot{q}_2 = \dot{x}_2$$
$$\dot{q}_3 = 3\dot{x}_2 - 3\dot{x}_1 + 11x_2 - 11x_1 + 2F_1(t) + 5F_2(t)$$
$$\dot{q}_4 = -4\dot{x}_2 + 4\dot{x}_1 - 25x_2 + 25x_1 + F_1(t) + 5F_2(t)$$

$\dot{q} = A\underline{q} + B\underline{u}$

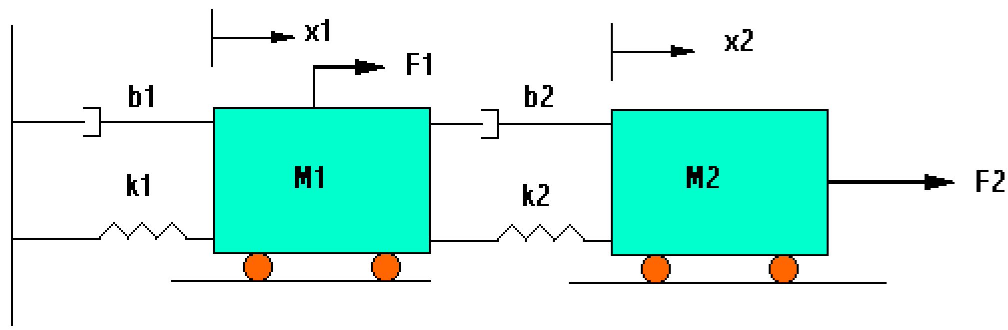
$$\dot{q} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 11 & -3 & 3 \\ 25 & -25 & 4 & -4 \end{bmatrix} \underline{q} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 5 \\ 1 & 5 \end{bmatrix} \underline{u}$$

Output Equation

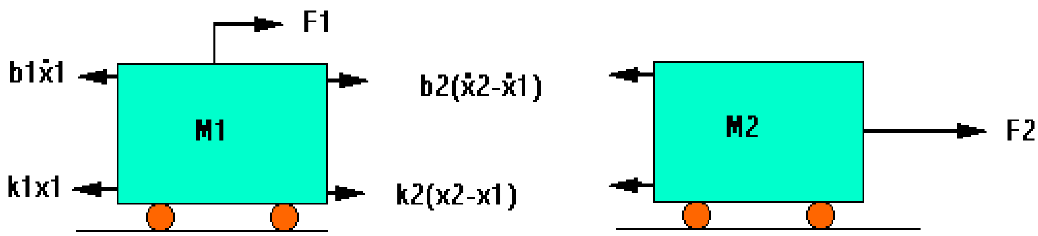
$\underline{y} = C\underline{q} + D\underline{u}$

$$\underline{y} = \begin{bmatrix} 3 & 0 & 0 & 5 \\ 0 & 0 & 4 & 0 \end{bmatrix} \underline{q} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \underline{u}$$

Example # 5 With Multiple Dependant Variables



FBD



$$m_1\ddot{x}_1 = k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) + F_1 - k_1x_1 - b_1\dot{x}_1$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) + F_2$$

F_1 and F_2 are the inputs and output is

$$y_1 = k_1x_1 + b_1\dot{x}_1$$
$$y_2 = k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1)$$

Define States

$$\underline{q} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad \text{and} \quad \underline{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

State Equations

$$\dot{q}_1 = \dot{x}_1$$
$$\dot{q}_2 = \dot{x}_2$$
$$\dot{q}_3 = \frac{k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) + F_1 - k_1 x_1 - b_1 \dot{x}_1}{m_1}$$
$$\dot{q}_4 = \frac{-k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) + F_2}{m_2}$$

$\dot{q} =$ A$\underline{q}$ + B$\underline{u}$

$$\dot{q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_2 - k_1}{m_1} & \frac{k_2}{m_1} & \frac{-b_2 - b_1}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \frac{b_2}{m_2} & \frac{-b_2}{m_2} \end{bmatrix} \underline{q} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \underline{u}$$

Output Equation

$\underline{y}$ = C$\underline{q}$ + D$\underline{u}$

$$\underline{y} = \begin{bmatrix} k_1 & 0 & b_1 & 0 \\ -k_2 & k_2 & -b_2 & b_2 \end{bmatrix} \underline{q} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}$$

Chapter

3

CONTINUOUS CONTROL

MAE 443

Routh Hurwitz

Routh - Hurwitz

Routh-Hurwitz is the system whose characteristic equation is:

$$a_n s^n + a_{n-1} s^{n-1} + \dots\dots\dots a_1 s + a_0 = 0$$

Stable? Yes if $Re(s_i) < 0$ for all roots s_i, $i=1, 2, \dots n$

No if $Re(s_i) > 0$ for any root

1^{st} Order: s_i easy to find roots

2^{nd} Order: $s_{1,2}$ quadratic, easy to find

3^{rd} Order or Higher: much more difficult to use, computer can easily solve

Bigger Issue: What if some a_i's are not yet known, such as design variables? This is why we use Routh-Hurwitz to determine range for stability.

Routh-Hurwitz

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}
s^{n-2}	$\frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = b_1$	$\frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} = b_2$	$\frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}} = b_3$	$\vdots$
s^{n-3}	$\frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}$	$\frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}$	$\vdots$	$\vdots$
s^0	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Interpretation of the Routh-Hurwitz Table:

The Number of “sign changes” in the 1^{st} column = number of roots with Re > 0

$\therefore$ Stable system has <u>same</u> <u>sign</u> elements in entire first row.

(Note: A “zero” in the first column means Re ≥ 0, which is normally “not good enough” due to either instability or only marginal stability.)

Note: “Pre-Check”- if all a_i’s (coefficients don’t have the same sign, system is unstable and has roots Re > 0

EXAMPLES

Example # 1 Given: $s^2 + 2s + 1$

Routh Table

s^2	1	1
s^1	2	0
s^0	$\frac{2-0}{2}=1$	

Example # 2 Given: $3s^2 + 2s - 6$

"Unstable, <u>Sign change</u> in the Characteristic Equation (- 6 term)

Example # 3 Given: $s^5 + 3s^4 - 2s^3 + 6s^2 + s - 3$

"Unstable", <u>Sign change</u> in the Characteristic Equation (Routh Table Anyways)

Routh Table

s^5	1	-2	1
s^4	3	6	-3
s^3	$\frac{(3)(-2)-(6)(1)}{3}=-4$	$\frac{(3)(1)-(-3)(1)}{3}=2$	0
s^2	$\frac{(-4)(6)-(2)(3)}{-4}=7.5$	$\frac{(-4)(-3)-(0)(3)}{-4}=-3$	
s^1	$\frac{(7.5)(2)-(-3)(-4)}{7.5}=0.4$		
s^0	-3		

Example # 4 Given: $s^3 + s^2 + (b-1)s + (k-1)$

Find Values of b (damping) and k (stiffness) to make stable

Initial Conditions: b > 1 and k > 1

Routh Table

s^3	1	(b-1)
s^2	1	(k-1)
s^1	$\frac{(1)(b-1)-(1)(k-1)}{1} = b-k$	
s^0	k-1	

Additional Conditions: k > 1 and b > k

Overall Conditions: 1) b > 1

2) k > 1

3) b > k

Example # 5 Given: $3s^4 + 2s^3 + ks^2 + 10s + 5$

Find Values of k to make system stable

Initial Conditions: $k > 0$

Routh Table

s^4	3	k	5
s^3	2	10	0
s^2	$\frac{((2)(k)-(3)(10)}{2} = k-15$	$\frac{(2)(5)-(0)(3)}{2} = 5$	
s^1	$\frac{(k-15)(10)-(2)(5)}{k-15} = \frac{10k-160}{k-15}$		
s^0	5		

Additional Conditions: $k > 15$ and $k > 16$

Overall Conditions: 1) $k > 0$

2) $k > 15$

3) $k > 16$

Most Dominating Condition for Stability is $k > 16$

AXIS-SHIFT

Sometimes we need to find a system to be stable within a certain time constant or settling time. Replace the characteristic equation with the given requirements. Refer to examples

EXAMPLES

Example # 1 Find the conditions on the design variable k to insure that the "settling time" is less than 1 second, if the system is: $s^3 + ks^2 + 10s + 4 = 0$

Reminder that S.T.= $\frac{4}{a}$ $\therefore$ $1 = \frac{4}{a}$, a = 4

$\tau = \frac{1}{a} = 0.25$

Axis – Shift is defined as $\left(s - \frac{1}{\tau} \right)$ $\therefore$ Replace Characteristic Equation with (s - 4) and do Routh Table to find stability condition, if they exist!

$(s – 4)^3 + k(s – 4)^2 +10(s – 4) + 4 = 0$

Expanded: $s^3 + (k-12)s^2 + (58 – 8k)s + (16k – 100) = 0$

Initial Conditions: $k > 12$ and $k < 7.25$

Impossible Conditions to Occur, Therefore no need for Routh Table!!

Example # 2 Find all values (if any) of the parameter k such that the system whose system is defined as: s4 + $8s^3 + ks^2 + 32s + 16 = 0$ and has a "time constant" less than 1.

$\tau \leq 1$

Axis – Shift is defined as $\left(s - \frac{1}{\tau}\right)$ $\therefore$ Replace Characteristic Equation with (s - 1) and do Routh Table to find stability condition, if they exist!

$$(s – 1)^4 + 8(s – 1)^3 + k(s – 1)^2 +32(s – 1) + 16 = 0$$

Expanded: $s^4 + 4s^3 + (k-18)s^2 + (52 – 2k)s + (k – 23) = 0$

Initial Conditions: k > 18 and k < 26 and k > 23

Routh Table

s^4	1	(k-18)	(k-23)
s^3	4	(52-2k)	0
s^2	$\frac{(4)(k-18)-(1)(52-2k)}{4} = \frac{3k-62}{2} = a$	$\frac{(4)(k-23)-(1)(0)}{4} = k-23$	
s^1	$\frac{(a)(52-2k)-(4)(k-23)}{a} = \frac{\left(\frac{3k-62}{2}\right)(52-2k)-(4k-92)}{\frac{3k-62}{2}}$		
s^0	k-23		

Additional Conditions: k > 20.66 and k > 23 and (s^1 term: k > 20 and k < 25.33)

Most Dominating Condition for Stability is k > 23 and k < 25.33

Chapter

4

System Identification

Curve Fitting by Least Squares

Given a set of points: experimental data, tabular data, etc.

Fit a curve (surface) to the points so that we can easily evaluate f(x) at any x of interest.

If x within data range —> interpolating (generally safe)

If x outside data range—> extrapolating (often dangerous)

Two main methods to be covered:

1. Least-Squares Regression
   - Function is "best fit" to data.
   - Does not necessarily pass through points.
   - Used for scattered data (experimental)
   - Can develop empirical models for analysis/design.
2. Interpolation
   - Function passes through all (or most) points.
   - Interpolates values of well-behaved (precise) data or for geometric design.

Curve Fitting by Least-Squares Regression

Objective: Obtain a low order curve (surface) which "best fits" to data.

Note: Because the order of the fitting curve is less than the number of data points, the curve (or surface) will not pass through all the points.

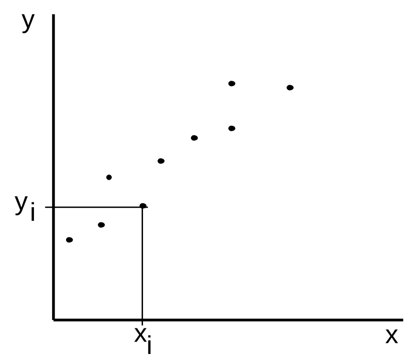
We need a *consistent* criterion for determining the "best fit."

Typical Usage:

Scattered (experimental) data

Develop empirical models for analysis/design.

Least-Squares Regression



1. In laboratory, apply **x** (independent variable), measure **y** (dependent variable).
2. Plot data and examine relationship.
3. Develop a "model" – an approximate relationship between y and x, such as **y = m x + b**
4. Use the model to predict or estimate **y** for any given **x**.
5. "Best fit" of the data requires:
   - Optimal way of finding parameters (e.g., slope and intercept of a straight line).
   - Perhaps optimize the selection of the model form (i.e., linear, quadratic, exponential,….etc)
   - That the magnitudes of the residual errors do not vary in any systematic fashion. [In statistical applications, the residual errors should be independent and identically distributed.]

Curve Fitting by Least-Squares Regression

Given: n data points: $(x_1,y_1),(x_2,y_2),\ldots(x_n,y_n)$

Obtain: "Best fit" curve --

$$f(x) = a_0\, Z_0(x) + a_1\, Z_1(x) + a_2\, Z_2(x) + \ldots + a_m\, Z_m(x)$$

a_i's are *unknown* parameters of model

Z_i's are *known* functions of x.

We will focus on two of the many possible types of regression models:

• Simple Linear Regression

$Z_0(x) = 1$ and $Z_1(x) = x$

• General Polynomial Regression

$Z_0(x) = 1,\ Z_1(x) = x,\ Z_2(x) = x^2,\ \ldots,\ Z_m(x) = x^m$

General Procedure

For i^{th} data point, (x_i,y_i) we find the set of coefficients for which:

$$y_i = a_0\, Z_0(x_i) + a_1\, Z_1(x_i) \ldots\ldots + a_m\, Z_m\,(x_i) + e_i$$

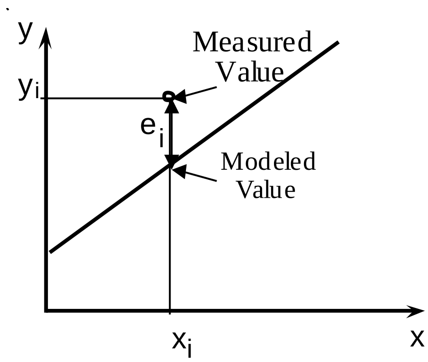
where e_i is the ***residual error*** = *the difference between reported value and model*

$$e_i = y_i - a_0 Z_0\,(x_i) - a_2 Z_2\,(x)_i - \ldots - a_m Z_m\,(x_i)$$

Our "best fit" will minimize the ***total sum of the squares*** of the residuals:

$$S_r = \sum_{i=1}^{n} e_i^{\,2} = \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} a_j Z_j(x_i) \right)^2$$

$$= \sum_{i=1}^{n} \left(y_i - a_0 Z_0(x_i) - a_1 Z_1(x_i) - a_2 Z_2(x_i) - \cdots - a_m Z_m(x_i) \right)^2$$



To minimize this with respect to the unknowns a_0, a_1, … a_m take derivative of S_r with respect to the unknowns and set equal to zero:

$$\frac{\partial S_r}{\partial a_0} = -2\sum_{i=1}^{n} Z_0(x_i)\Big(y_i - a_0 Z_0(x_i) - a_1 Z_1(x_i) - a_2 Z_2(x_i) - \cdots - a_m Z_m(x_i)\Big) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum_{i=1}^{n} Z_1(x_i)\Big(y_i - a_0 Z_0(x_i) - a_1 Z_1(x_i) - a_2 Z_2(x_i) - \cdots - a_m Z_m(x_i)\Big) = 0$$

$$\vdots$$

$$\frac{\partial S_r}{\partial a_m} = -2\sum_{i=1}^{n} Z_m(x_i)\Big(y_i - a_0 Z_0(x_i) - a_1 Z_1(x_i) - a_2 Z_2(x_i) - \cdots - a_m Z_m(x_i)\Big) = 0$$

In Linear Algebra form:

$\{Y\} = [Z]\{A\} + \{E\}$ or $\{E\} = \{Y\} - [Z]\{A\}$

where

$\{E\}$ and $\{Y\}$ are n x 1

$[Z]$ is n x m+1

$\{A\}$ is (m+1) x 1

n=# of points

(m+1)=# of unknowns

$\{E\}^T = [e_1 \ e_2 \ \ldots \ e_n]$,

$\{Y\}^T = [y_1 \ y_2 \ \ldots \ y_n]$,

$$Z = \begin{bmatrix} Z_0 x_1 & Z_0 x_1 & \cdots & Z_m x_1 \\ Z_0 x_2 & Z_1 x_2 & \cdots & Z_m x_2 \\ \vdots & \vdots & \ddots & \vdots \\ Z_0 x_n & Z_1 x_n & \cdots & Z_m x_n \end{bmatrix}$$

$\{A\}^T = [a_0 \ a_1 \ a_2 \ \ldots \ a_m]$

$$S_r = \{E\}^T\{E\} = (\{Y\}-[Z]\{A\})^T (\{Y\}-[Z]\{A\})$$

$$= \{Y\}^T\{Y\} - \{A\}^T[Z]^T\{Y\} - \{Y\}^T[Z]\{A\} + \{A\}^T[Z]^T[Z]\{A\}$$

$$= \{Y\}^T\{Y\} - 2\,\{A\}^T[Z]^T\{Y\} + \{A\}^T[Z]^T[Z]\{A\}$$

Setting $\dfrac{\partial S_r}{\partial a_i} = 0$ for i=1,n yields: $0 = 2\,[Z]^T[Z]\{A\} - 2\,[Z]^T\{Y\}$

or

$$[\mathbf{Z}]^T[\mathbf{Z}]\{\mathbf{A}\} = [\mathbf{Z}]^T\{\mathbf{Y}\}$$

This is the general form of ***Normal Equations.***

They provide (m+1) equations in (m+1) unknowns.

(Note that we end up having to solve a system of linear algebraic equations.)

Simple Example -- Linear Regression (m = 1)

Given: n data points, $(x_1,y_1),(x_2,y_2)\ldots(x_n,y_n)$ w/ $n>2$

Obtain: "Best fit" linear curve: $f(x) = a_0 + a_1 x$ from the n equations:

$$y_1 = a_0 + a_1 x_1 + e_1$$

$$y_2 = a_0 + a_1 x_2 + e_2$$

$$\vdots$$

$$y_n = a_0 + a_1 x_n + e_n$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}$$

Or, in matrix form: $[Z]^T[Z]\,\{A\} = [Z]^T\{Y\}$

Upon multiplying, the matrices become:

Normal Equations for Linear Regression

(This form works for spreadsheets.)

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{Bmatrix}$$

Solving for {a}:

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \qquad a_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - a_1\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$$

or better, $$a_1 = \frac{\sum_{i=1}^{n}\left[\; y_i - y_{avg} \quad x_i - x_{avg} \;\right]}{\sum_{i=1}^{n} \; x_i - x_{avg} \;\; ^2}$$ which is numerically more stable, and

$$a_0 = y_{avg} - a_1 x_{avg}$$

Common Nonlinear Relations

Objective: Use linear equations for simplicity

Remedy: Transform data into linear form and perform regressions

Given: data which appears as

(1) exponential-like curve $\mathbf{y = a_1 e^{b_1 x}}$

(e.g., population growth, radioactive decay, transmission line loss, etc.)

Can use $\ln(y) = \ln(a_1) + b_1 x$

(2) Power-like curve $\mathbf{y = a_2 x^{b_2}}$

Can use $\ln(y) = \ln(a_2) + b_2 \ln(x)$

(3) Saturation growth-rate curve: $y = \dfrac{a_3 x}{b_3 + x}$

population growth under limiting conditions

Can use $1/y = a_3' + b_3'/x$

Be careful about the implied distribution of the errors.

Major Points in Least-Squares Regression

1. In all regression models one is solving an over determined system of equations, i.e., more equations than unknowns.

2. How good is the fit?

Often based on a **coefficient of determination, r^2**.

r^2 : Compares the average spread of the data about the ***regression line*** to the spread of the data about the ***mean***. Spread the data around the regression line:

$$S_r = \sum e_i^2 = \sum (y_i - y'_i)^2$$

Spread of the data around the mean:

$$S_t = \sum (y_i - \bar{y})^2$$

The **coefficient of determination** describes how much of variance is "explained" by the regression equation.

$$\mathbf{r^2 = \frac{S_t - S_r}{S_t}}$$

Using ***coefficient of determination,*** r^2

- Want r^2 close to 1.0.
- Doesn't work if models have different numbers of parameters.
- Be careful when using different transformations. Compare the r^2 using the original, not the transformed values.

Precision of the Regression Estimate

If the spread of the points around the line is of similar magnitude along the entire range of the data, and

Then one can use

$$S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$ = **standard error of the estimate**

(standard deviation in y)

to describe the precision of the regression estimate.

Examples: Exponential Decaying Sinusoid

Model:

$$Ae^{at}\sin(bt+\phi)=f(t)$$

Data = $\underline{y}$

Unknowns: $\underline{x}$ = $\begin{Bmatrix} A \\ a \\ b \\ \phi \end{Bmatrix}$

$$\frac{\partial f}{dx}=\begin{bmatrix} \frac{\partial f_1}{dA} & \frac{\partial f_1}{da} & \frac{\partial f_1}{db} & \frac{\partial f_1}{\partial \phi} \\ \frac{\partial f_2}{dA} & \frac{\partial f_2}{da} & \frac{\partial f_2}{db} & \frac{\partial f_2}{\partial \phi} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial f_n}{dA} & \frac{\partial f_n}{da} & \frac{\partial f_n}{db} & \frac{\partial f_n}{\partial \phi} \end{bmatrix}$$

Where the Matrix is made up of the partials

$$\frac{\partial f}{\partial A}=e^{at}\sin(bt+\phi)$$

$$\frac{\partial f}{\partial a}=Ate^{at}\sin(bt+\phi)$$

$$\frac{\partial f}{\partial b}=Ate^{at}\cos(bt+\phi)$$

$$\frac{\partial f}{\partial \phi}=Ae^{at}\cos(bt+\phi)$$

Fourier Example:

Model:

$$A\sin\varpi_1 t + B\sin\varpi_2 + C\cos\varpi_1 t + D\cos\varpi_2 = f(t)$$

$$\frac{\partial f}{\partial A} = \sin\varpi_1 t$$

$$\frac{\partial f}{\partial C} = \cos\varpi_1 t$$

$$\frac{\partial f}{\partial B} = \sin\varpi_2 t$$

$$\frac{\partial f}{\partial D} = \cos\varpi_2 t$$

$$\frac{\partial f}{\partial \varpi_1} = t(A\cos\varpi_1 t - C\sin\varpi_1 t)$$

$$\frac{\partial f}{\partial \varpi_2} = t(B\cos\varpi_2 t - D\sin\varpi_2 t)$$

MATLAB CODE

Now using Matlab and running a program, we will be able to

solve the non-linear approximation.

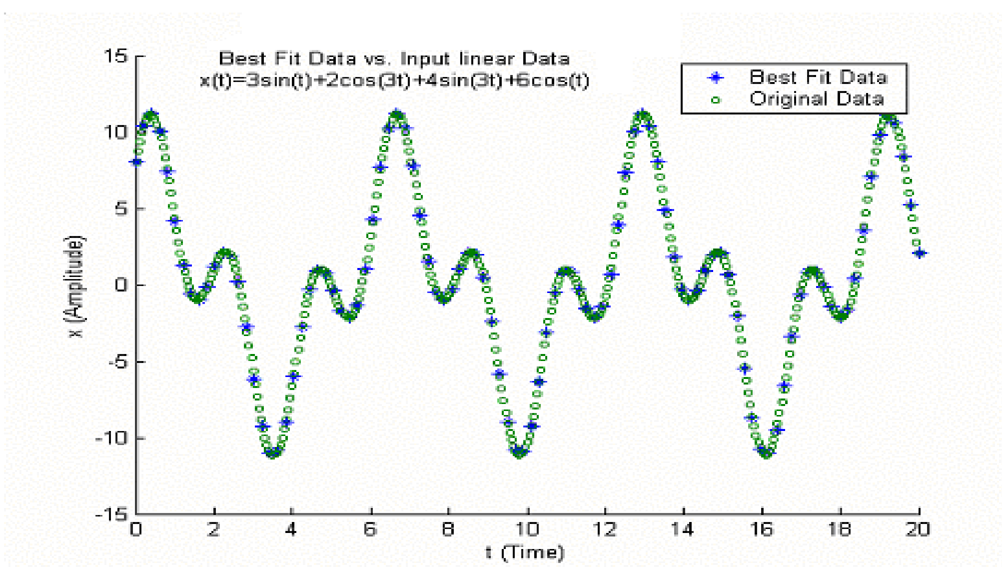
% Here is the M-File to solve the problem above

```
function solve(t,y,x0)
x=x0;
for k=1:20
    for i=1:size(t)
        pfpx(i,1)=sin(x(5)*t(i));
        pfpx(i,2)=cos(x(5)*t(i));
        pfpx(i,3)=sin(x(6)*t(i));
        pfpx(i,4)=cos(x(6)*t(i));
        pfpx(i,5)=t(i)*(x(1)*cos(x(5)*t(i))-x(2)*sin(x(5)*t(i)));
        pfpx(i,6)=t(i)*(x(3)*cos(x(6)*t(i))-x(4)*sin(x(6)*t(i)));

f(i,1)=x(1)*sin(x(5)*t(i))+x(2)*cos(x(5)*t(i))+x(3)*sin(x(6)*t(i))+x(4)*
cos(x(6)*t(i));
    end
    dy=y-f;
    dx=inv(pfpx'*pfpx)*pfpx'*dy;
    phi=dy'*dy

    x=x+dx
    plot(t,y,'*',t,dy,'o')
    pause(.5)
end

%solve.m must be in the working directory when loading the data
%Here are the commands you do at the prompt
%load data
%t=data(:,1);
%y=data(:,2);
%x0=[10 0 0 20 3 2]';  Initial guess for 1-6 unknowns
%solve(t,y,x0)
```



Chapter

5

CONTINUOUS CONTROL

MAE 443

Transfer Functions

Transfer Functions

Reminder:

We live in Time Domain: t

Know we want to convert time to Laplace Domain

$$L(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Used to solve Linear, Time, Invariant ODE's

Transfer Function $= \dfrac{L(Output)}{L(Input)}$, I.C.'s = 0

Note: One Transfer Function for each input/output combination in a system, for example, 3 inputs + 4 outputs = 12 TF's. Also in systems with more than one input and/or more than one output, find each TF by setting the other inputs and outputs to zero.

$$TF = \frac{N(s)}{D(s)} = \frac{Num}{Den}\ \frac{polynomial}{polynomial}$$

"Zero" = root of N(s)

"Pole" = root of D(s)

Example: $m\ddot{x} + b\dot{x} + kx = f(t) \begin{Bmatrix} output, y = x(t) \\ input, u = f(t) \end{Bmatrix}$

$$\therefore\ TF = \frac{Y(s)}{U(s)} = \frac{X(s)}{F(s)} \text{ with I.C.'s} = 0$$

Take Laplace Transform: $ms^2X(s) + bsX(s) + kX(s) = F(s)$

$$\therefore\ \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \begin{cases} Zero = None \\ Pole = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m} \end{cases}$$

Suppose output = y(t) = $\ddot{x} + \dot{x}$

$$TF = \frac{Y(s)}{U(s)} = \frac{s^2 X(s) + sX(s)}{F(s)} = \frac{s^2 + s}{ms^2 + bs + k} \begin{cases} Zero = 0, -1 \\ Pole = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m} \end{cases}$$

Suppose there were two inputs:

$$m\ddot{x} + b\dot{x} + kx = f(t) + 2g(t) \begin{Bmatrix} u_1 = f \\ u_2 = g \end{Bmatrix}$$

Let y = x(t)

$$\therefore \quad TF_1 = \frac{X(s)}{U(s)} = \frac{1}{ms^2 + bs + k} \quad Assume, g = 0$$

$$TF_2 = \frac{X(s)}{U(s)} = \frac{2}{ms^2 + bs + k} \quad Assume, f = 0$$

Some Transfer Function Properties

1) All TF's for a system have the same denominator and the roots of the denominator (= poles) are equal to the roots of the system's Characteristic Equation.

2) $\lim_{s \to 0} (TF) = y(t \to \infty)$ for $u = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

3) $\lim_{s \to \infty} (TF) = y(t_0^{\,+}) - y(t_0^{\,-})$ for $u = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

4) TF(s = $i\omega$) = "Frequency Response Function" = FRF

FRF = Complex Function of ω = Re(ω) + iIm(ω)

Refer to Chapter on FRF for additional information☺

State Space Transfer Function

As usual, state space offers a very efficient way to determine all of the system's transfer functions in a standard calculation

State Space Model:

$\underline{\dot{x}} = A\underline{x} + B\underline{u}$

$\underline{y} = C\underline{x} + D\underline{u}$

$\underline{x} = n \times 1$

$\underline{y} = m \times 1$

$\underline{u} = r \times 1$

Take Laplace Transform:

$s\underline{x}(s) - x(0) = A\underline{x}(s) + B\underline{u}(s)$

$s\underline{x}(s) - A\underline{x}(s) = B\underline{u}(s)$ (x (0) = 0 for TF)

$[sI - A]\underline{x}(s) = B\underline{u}(s)$

$\underline{x}(s) = [sI - A]^{-1} B\, \underline{u}(s)$

Now: $\underline{Y}(s) = C\underline{x}(s) + D\underline{u}(s)$

$\underline{Y}(s) = \{C[sI - A]^{-1} B + D\}\, u(s)$

Where $\{C[sI - A]^{-1} B + D\}$ is the Transfer Function Matrix.

EXAMPLES

Example # 1 Find the Transfer Function $\frac{X(s)}{F(s)}$ for the system modeled by

$3\ddot{x}+2\dot{x}+4x=5f(t)$

Answer: $\frac{X}{F}=\frac{Output}{Input}=\frac{5}{3s^2+2s+4}$

Example # 2 Find the Transfer Function $\frac{Y(s)}{F(s)}$ if the output is $y(t)=4\dot{x}+2x$, for the

system: $\ddot{x}+6\dot{x}+5x=f(t)+\frac{df(t)}{dt}$

Answer: $\frac{X}{F}=\frac{s+1}{s^2+6s+5}$ $\quad\therefore\quad$ $\frac{Y}{F}=\frac{(s+1)(4s+2)}{s^2+6s+5}=\frac{4s^2+6s+2}{s^2+6s+5}$

Example # 3 Consider the Transfer Function $\frac{Y(s)}{F(s)}=\frac{s^2+2s}{3s^2+4s+5}$

QA. Find $y(t=0^+)$ *immediately after a step input is applied, f(t = 0) = 6, if both y = 0 and f = 0 prior to the step input.*

Answer: $6*\lim_{s\to\infty}\frac{Y}{F}=6*\frac{1}{3}=2$

QB. Find $y(t\to\infty)$ *for the same conditions in previous problem.*

Answer: $6*\lim_{s\to 0}\frac{Y}{F}=0$

Example # 4 Consider the Transfer Function $\frac{Y(s)}{F(s)} = \frac{s+3}{s^2+2s+25}$

QA. Find $y(t=0^+)$ *immediately after a step input is applied, f(t = 0) = 1, if both y = 0 and f = 0 prior to the step input.*

Answer: $1 * \lim_{s\to\infty} \frac{Y}{F} = 1 * \frac{s}{s^2} = 0$

QB. Find $y(t \to \infty)$ *for the same conditions in previous problem.*

Answer: $1 * \lim_{s\to 0} \frac{Y}{F} = \frac{3}{25}$

QC. What are the "Zeroes" of the system:

Answer: s = - 3

QD. What are the "Poles" of the system:

Answer: $s_{1,2} = -1.0 \pm 4.89i$

QE. Find the systems natural frequency?

Answer: $s^2 + 2s + 25$

$\omega_n = \sqrt{25} = 5$ rad/sec

QF. Find the systems damping ratio?

Answer: $2\xi\omega_n = 2 \quad \therefore \ \xi = 0.2$

QF. What is the time constant?

Answer: $\tau = \frac{1}{\xi\varpi_n} = 1$ second

Example # 5 Write the expression for the Transfer Function which has zeroes at (-6) and (-4), and poles at (-2 ± 4*i*)

$$TF = \frac{(s+6)(s+4)}{(s+5)(s+2+4i)(s+2-4i)} = \frac{(s+6)(s+4)}{(s+5)(s^2+4s+20)}$$

Example # 6 Write the differential model for a system whose Transfer Function $\frac{X(s)}{F(s)}$ is

$$\frac{7}{s^2+4s+3}$$

Answer: $\ddot{x}+4\dot{x}+3x = 7f(t)$

Example # 7 Write the differential model for a system whose Transfer Function $\frac{X(s)}{F(s)}$ is

$$\frac{7s}{s^2+4s+3}$$

Answer: $\ddot{x}+4\dot{x}+3x = \frac{7df(t)}{dt}$

Chapter

6

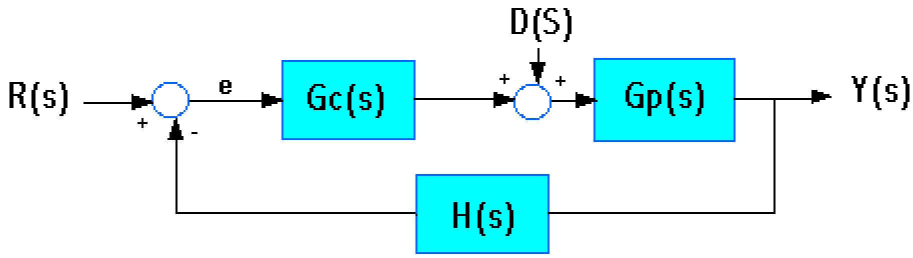
CONTINUOUS CONTROL

MAE 443

Simple System Diagrams

Simple System Diagrams

A Transfer Function in Laplace Domain is represented by a "Block"



R(s) is the "Reference", tells the system what to do.

e is the "Error"

Gc(s) is the "Controller"

D(s) is the "Disturbance" or outside environment

Gp(s) is the "Plant" or the system to be controlled

Y(s) is the "Output" of the system

H(s) is the "Observer" which monitors the input by using sensors and processors

Y=Gp[D + Gc(R – HY)]

(1 + GpGcH)Y =GpD + GpGcR

The Primary Transfer Function = PTF = $\dfrac{Y(s)}{D(s)} = \dfrac{GpGc}{1+GpGcH}$

The Disturbance Transfer Function = DTF = $\dfrac{Y(s)}{R(s)} = \dfrac{Gp}{1+GpGcH}$

System Goals or Control Design:

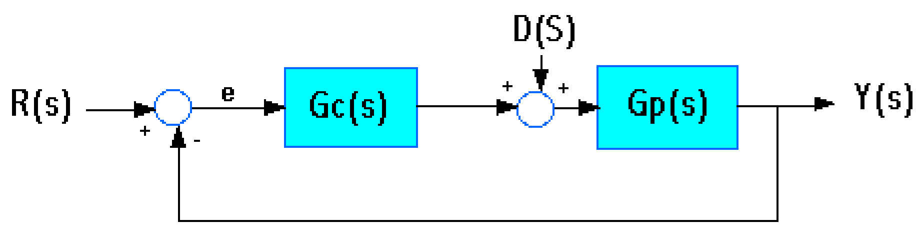
1) Stability: roots of the characteristic equation, denominator of TF

2) Speed: time constant (smaller the better), roots in the left hand plane

3) Accuracy: step response, $\lim_{s \to 0}(PTF)$

4) Robustness/Disturbance: step response, $\lim_{s \to 0}(DTF)$

Example of 1st Order Plant:



$$\text{Gp} = \frac{1}{s+1}$$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)Gc}{1+Gc\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{Gc}{(s+1)+Gc}$$

The Disturbance Transfer Function

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+Gc\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{1}{(s+1)+Gc}$$

Control Laws

Characteristics P-I-D

A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error. An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse. A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response. Effects of each of controllers Kp, Kd, and Ki on a closed-loop system are summarized in the table shown below.

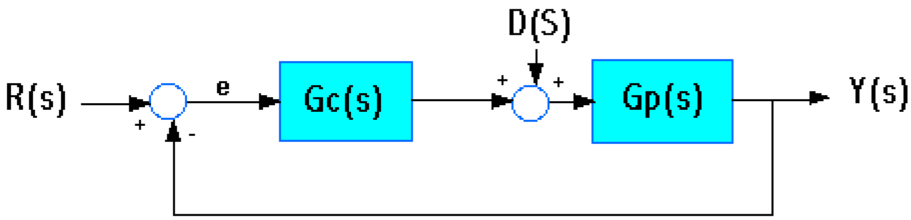
CL RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S-S ERROR
Kp	Decrease	Increase	Small Change	Decrease
Ki	Decrease	Increase	Increase	Eliminate
Kd	Small Change	Decrease	Decrease	Small Change

Note that these correlations may not be exactly accurate, because Kp, Ki, and Kd are dependent of each other. In fact, changing one of these variables can change the effect of the other two. For this reason, the table should only be used as a reference when you are determining the values for Ki, Kp and Kd.

Various Types of Controls

1) Proportional Control: "P"

Gc = Kp "Proportional control gain"



Let Gp = $\dfrac{1}{s+1}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)Kp}{1+Kp\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{Kp}{(s+1)+Kp}$$

The Disturbance Transfer Function

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+Kp\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{1}{(s+1)+Kp}$$

i) stability: Kp > -1

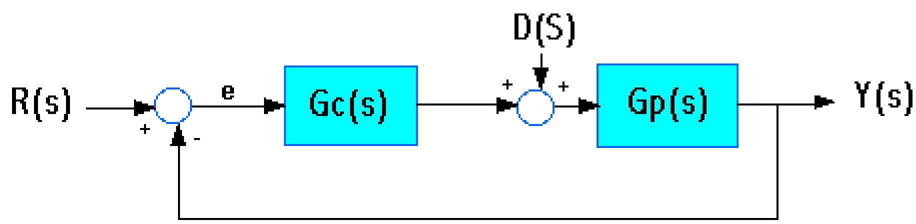
ii) speed: Kp as big as possible

iii) Accuracy: $\lim_{s\to 0}(PTF) = \dfrac{Kp}{1+Kp} = 0$

iv) Robustness: $\lim\limits_{s \to 0}(DTF) = \dfrac{1}{1+Kp} = 0$

2) Intergral Control: "I"

Gc = $\dfrac{Ki}{s}$ where Ki is the "Integral Gain"



Let Gp = $\dfrac{1}{s+1}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)\left(\frac{Ki}{s}\right)}{1+\left(\frac{Ki}{s}\right)\left(\frac{1}{s+1}\right)} * \frac{(s+1)*s}{(s+1)*s} = \frac{Ki}{s^2+s+Ki}$$

The Disturbance Transfer Function

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+\left(\frac{Ki}{s}\right)\left(\frac{1}{s+1}\right)} * \frac{(s+1)*s}{(s+1)*s} = \frac{s}{s^2+s+Ki}$$

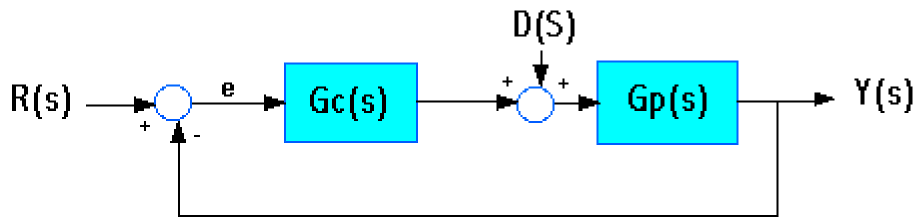
i) stability: Ki > 0 by doing Routh Table

ii) Accuracy: $\lim_{s\to 0}(PTF) = \dfrac{Ki}{s^2 + s + Ki} = 1$

iii) Robustness: $\lim_{s\to 0}(DTF) = \dfrac{s}{s^2 + s + Ki} = 0$

3) Derivative Control: "D"

Gc = Kds



Let Gp = $\dfrac{1}{s+1}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)(Kds)}{1+(Kds)\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{Kds}{s+1+Kds}$$

The Disturbance Transfer Function

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+(Kds)\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{1}{s+1+Kds}$$

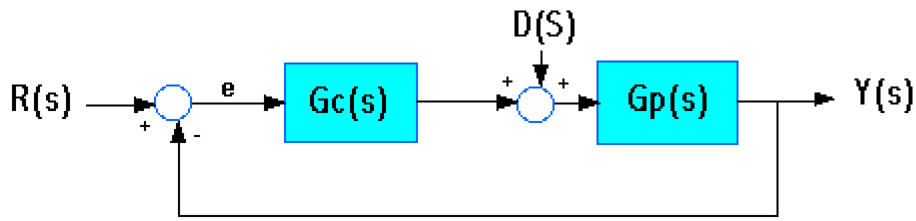
i) stability: Kd > -1 by doing Routh Table

ii) Accuracy: $\lim_{s\to 0}(PTF) = \dfrac{Kds}{s+1+Kds} = 0$

iii) Robustness: $\lim_{s\to 0}(DTF) = \dfrac{1}{s+1+Kds} = 1$

4) Proportional-Derivative Control: "P-D"

Gc = Kp + Kds



Let Gp = $\dfrac{1}{s+1}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$PTF = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)(Kp+Kds)}{1+(Kp+Kds)\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{Kp+Kds}{s+1+(Kp+Kds)}$$

The Disturbance Transfer Function

$$DTF = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+(Kp+Kds)\left(\frac{1}{s+1}\right)} * \frac{(s+1)}{(s+1)} = \frac{1}{s+1+(Kp+Kds)}$$

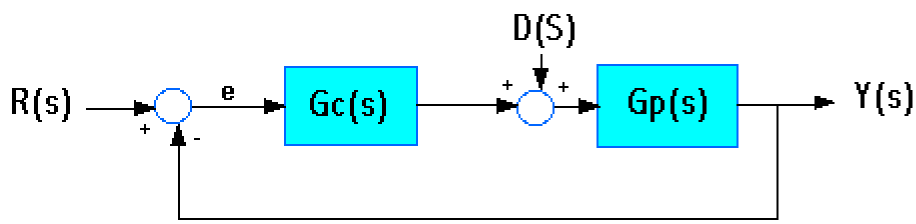
i) stability: Kd > -1 and Kp > -1 by doing Routh Table

ii) Accuracy: $\lim_{s\to 0}(PTF) = \dfrac{Kp + Kds}{s+1+(Kp+Kds)} = \dfrac{Kp}{1+Kp}$

iii) Robustness: $\lim_{s\to 0}(DTF) = \dfrac{1}{s+1+(Kp+Kds)} = \dfrac{1}{1+kp}$

5) Proportional-Integral Control: "P-I"

Gc $= Kp + \dfrac{Ki}{s}$



Let Gp = $\dfrac{1}{s+1}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)\left(Kp+\frac{Ki}{s}\right)}{1+\left(Kp+\frac{Ki}{s}\right)\left(\frac{1}{s+1}\right)} * \frac{(s+1)*s}{(s+1)*s} = \frac{Kps+Ki}{s^2+s+(Kps+Ki)}$$

The Disturbance Transfer Function

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+\left(Kp+\frac{Ki}{s}\right)\left(\frac{1}{s+1}\right)} * \frac{(s+1)*s}{(s+1)*s} = \frac{s}{s^2+s+(Kps+Ki)}$$

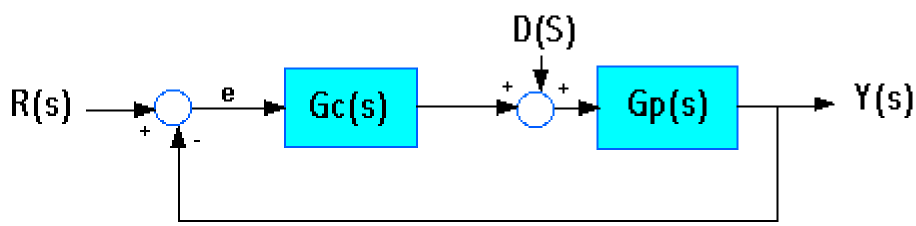
i) stability: Kp > -1 and Ki > 0 by doing Routh Table

ii) Accuracy: $\lim_{s\to 0}(PTF) = \dfrac{Kps + Ki}{s^2 + s + (Kps + Ki)} = \dfrac{Ki}{Ki} = 1$

iii) Robustness: $\lim_{s\to 0}(DTF) = \dfrac{s}{s^2 + s + (Kps + Ki)} = 0$

6) Proportional-Derivative-Integral Control: "P-I-D"

Gc = $Kp + Kds + \dfrac{Ki}{s}$



Let Gp = $\dfrac{1}{s+1}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

PTF =

$$\frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)\left(Kp + Kds + \frac{Ki}{s}\right)}{1+\left(Kp + Kds + \frac{Ki}{s}\right)\left(\frac{1}{s+1}\right)} * \frac{(s+1)*s}{(s+1)*s} = \frac{Kds^2 + Kps + Ki}{s^2 + s + (Kds^2 + Kps + Ki)}$$

The Disturbance Transfer Function

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s+1}\right)}{1+\left(Kds+Kp+\frac{Ki}{s}\right)\left(\frac{1}{s+1}\right)} * \frac{(s+1)*s}{(s+1)*s}$$

$$= \frac{s}{s^2+s+(Kds^2+Kps+Ki)}$$

i) stability: Kp > -1 and Ki > 0 and Kd > - 1 by doing Routh Table

ii) Accuracy: $\lim_{s\to 0}(PTF) = \frac{Kds^2+Kps+Ki}{s^2+s+(Kds^2+Kps+Ki)} = \frac{Ki}{Ki} = 1$

iii) Robustness: $\lim_{s\to 0}(DTF) = \frac{s}{s^2+s+(Kds^2+Kps+Ki)} = 0$

General tips for designing a PID controller

- When you are designing a PID controller for a given system, follow the steps shown below to obtain a desired response.
- Obtain an open-loop response and determine what needs to be improved
- Add a proportional control to improve the rise time
- Add a derivative control to improve the overshoot
- Add an integral control to eliminate the steady-state error
- Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response, then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

There are endless types of “Control Laws” in which one can achieve system goals.

Some other popular types are “Double Integral” and Double Derivative”

Gc = $\frac{Ki^2}{s^2}$ is “Double Integral Control”

Gc = Kd^2s^2 is “Double Derivative Control”

Chapter

7

CONTINUOUS CONTROL

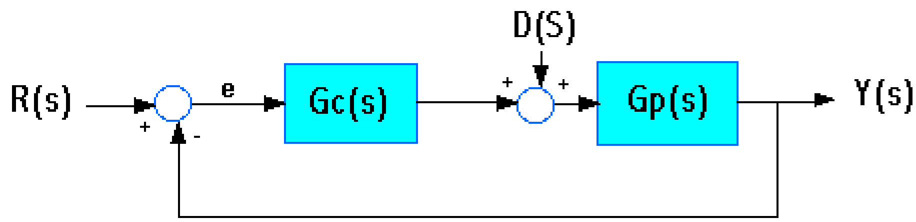
MAE 443

Complex System Diagrams

Complex System Diagrams

As seen in the previous chapter on block diagrams, these new block diagrams have special inputs/output combinations to achieve different system goals.

Basic



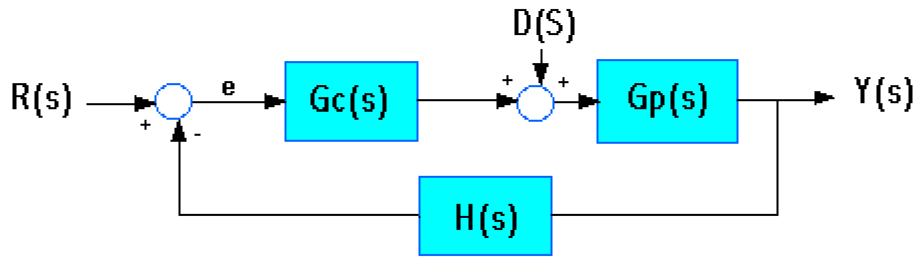
Y(s) = Gp[D + Gc(R - Y)]

(1 + GpGc)Y = GpD + GpGcR

PTF = $\frac{Y}{R} = \frac{GpGc}{(1+GpGc)}$

DTF = $\frac{Y}{D} = \frac{Gp}{(1+GpGc)}$

Basic with Observer



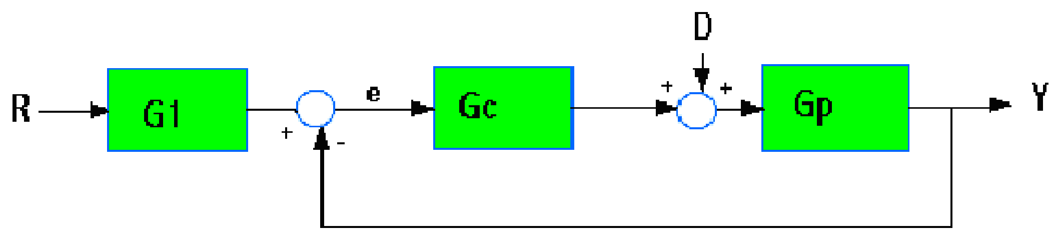
Y(s) = Gp[D + Gc(R - HY)]

(1 + GpGcH)Y = GpD + GpGcR

PTF = $\dfrac{Y}{R} = \dfrac{GpGc}{(1+GpGcH)}$

DTF = $\dfrac{Y}{D} = \dfrac{Gp}{(1+GpGcH)}$

Input Compensation



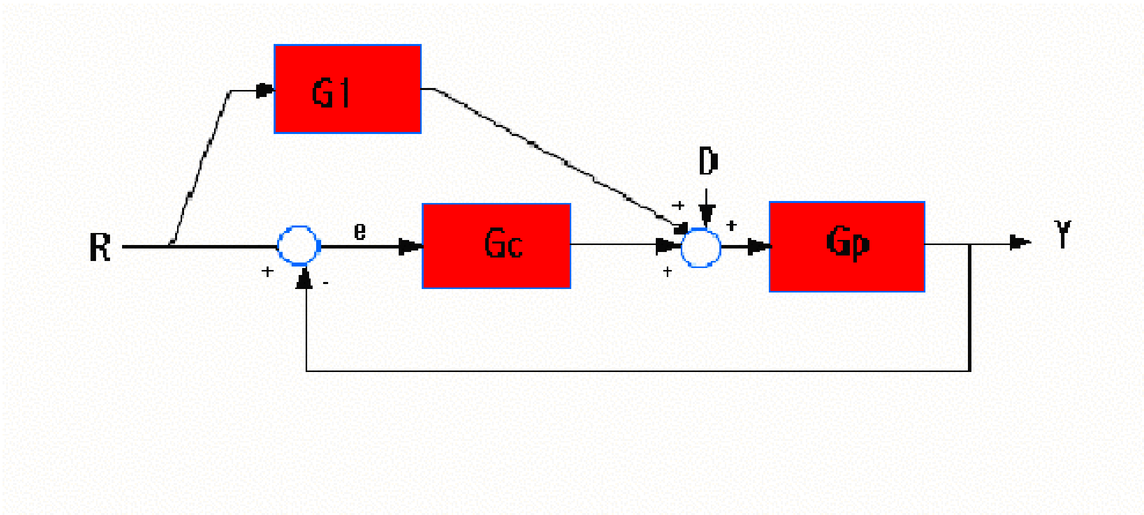
Y(s) = Gp[D + Gc(G1R - Y)]

(1 + GpGc)Y = GpD + GpGcG1R

PTF = $\frac{Y}{R} = \frac{GpGcG1}{(1+GpGc)}$

DTF = $\frac{Y}{D} = \frac{Gp}{(1+GpGc)}$

Feed Forward Compensation



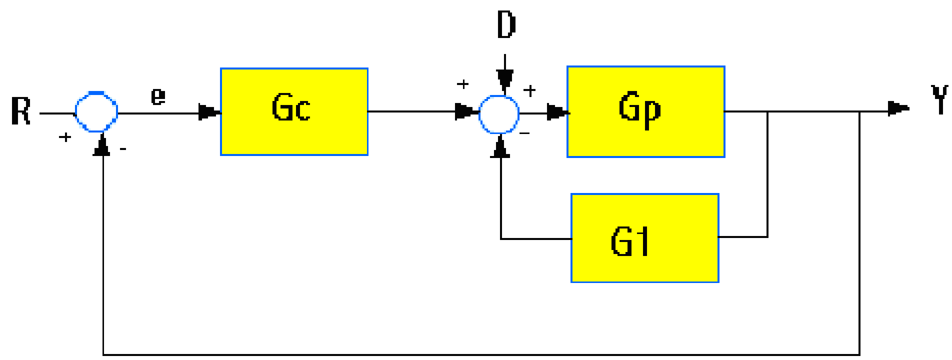
Y(s) = Gp[D + G1R + GcR - GcY]

(1 + GpGc)Y = GpD + Gp(G1 + Gc)R

PTF = $\frac{Y}{R} = \frac{Gp(G1+Gc)}{(1+GpGc)}$

DTF = $\frac{Y}{D} = \frac{Gp}{(1+GpGc)}$

Feed Back Compensation #1



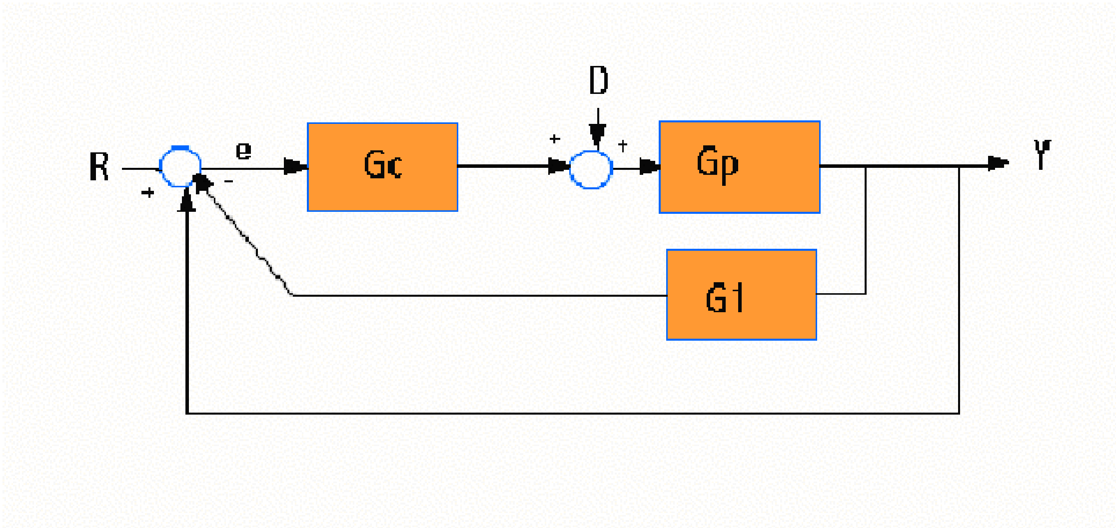
Y(s) = Gp[D - G1Y + GcR - GcY]

[1 + Gp(G1+Gc)]Y = GpD + GpGcR

PTF = $\frac{Y}{R} = \frac{GpGc}{1 + Gp(G1 + Gc)}$

DTF = $\frac{Y}{D} = \frac{Gp}{1 + Gp(G1 + Gc)}$

Feed Back Compensation #2



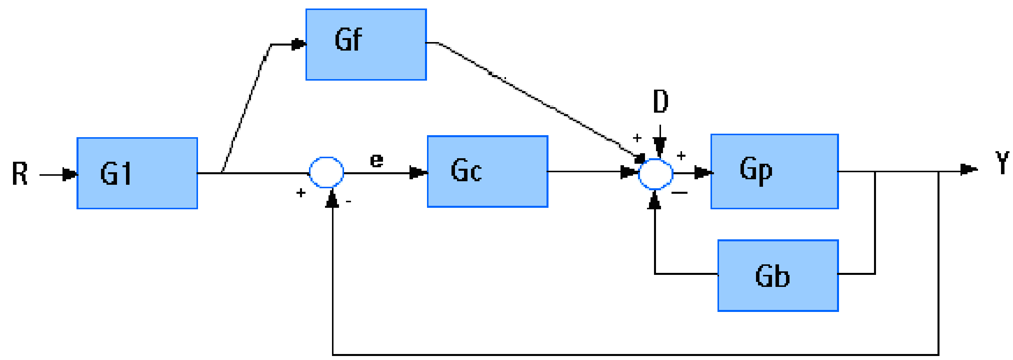
Y(s) = Gp[D + GcR – GcY – GcG1Y]

[1 + Gp(Gc + G1Gc)]Y = GpD + GpGcR

$$\text{PTF} = \frac{Y}{R} = \frac{GpGc}{1 + Gp(Gc + G1Gc)}$$

$$\text{DTF} = \frac{Y}{D} = \frac{Gp}{1 + Gp(Gc + G1Gc)}$$

Mixed System #1



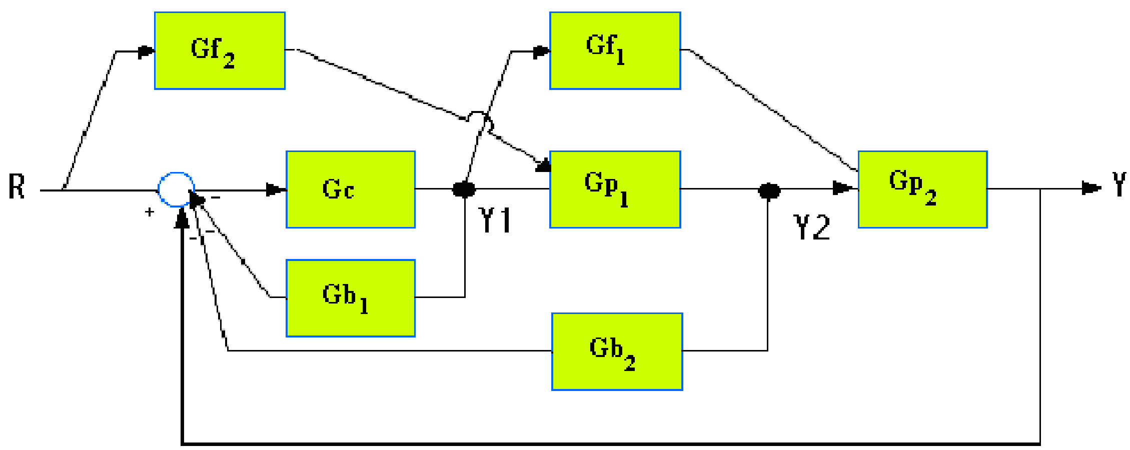
Y(s) = Gp[D - GbY + GfG1R + Gc(G1R – Y)]

[1 + Gp(Gb + Gc)]Y = GpD + GpGfG1R + GpGcG1R

$$\text{PTF} = \frac{Y}{R} = \frac{GpG1(Gf + Gc)}{1 + Gp(Gb + Gc)}$$

$$\text{DTF} = \frac{Y}{D} = \frac{Gp}{1 + Gp(Gb + Gc)}$$

Mixed System #2

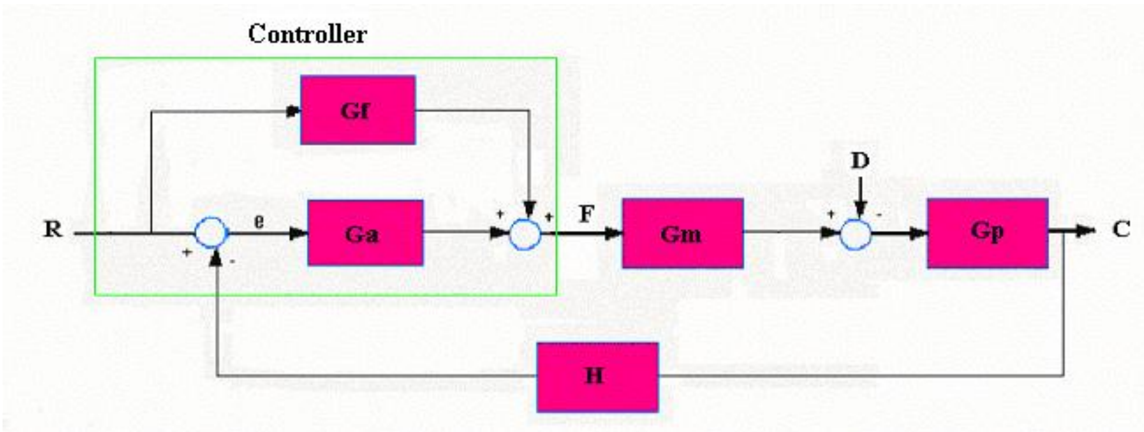


$Y_1 = Gc[R - Gb_1Y_1 - Gb_2Y_2 - Y]$ (1)

$Y_2 = Gp_1[Gf_2R + Y]$ (2)

Sub equation (2) into (1)

Mixed System # 3



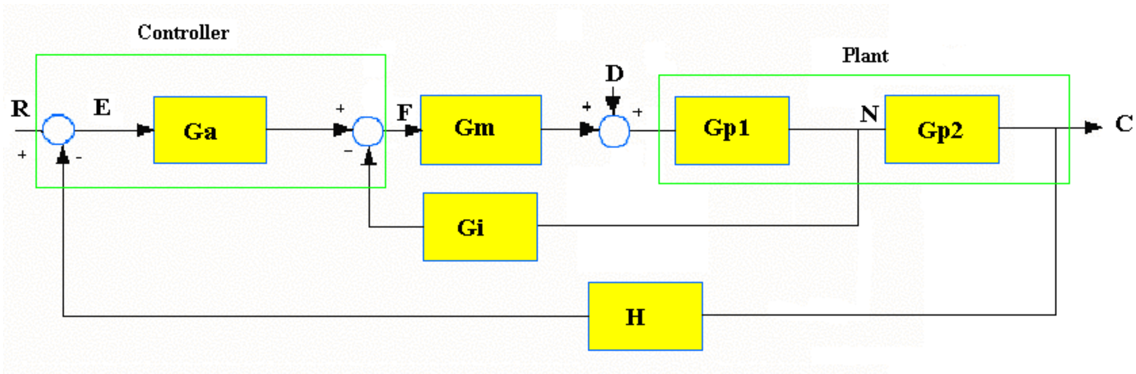
C(s) = Gp[- D + Gm(GfR + Ga(R – HC))]

[1 + GpGmGaH]C = - GpD + GpGmGfR + GpGmGaR

PTF = $\dfrac{C}{R} = \dfrac{GpGmGa}{1 + GpGmGaH}$

DTF = $\dfrac{C}{D} = \dfrac{-Gp}{1 + GpGmGaH}$

Mixed System # 4



N(s) = Gp1[D + Gm(- GiN) + GmGa(R – HC)] (1)

C(s) = Gp2N (2)

$\therefore$ Take Eqn (2) and put into (1)

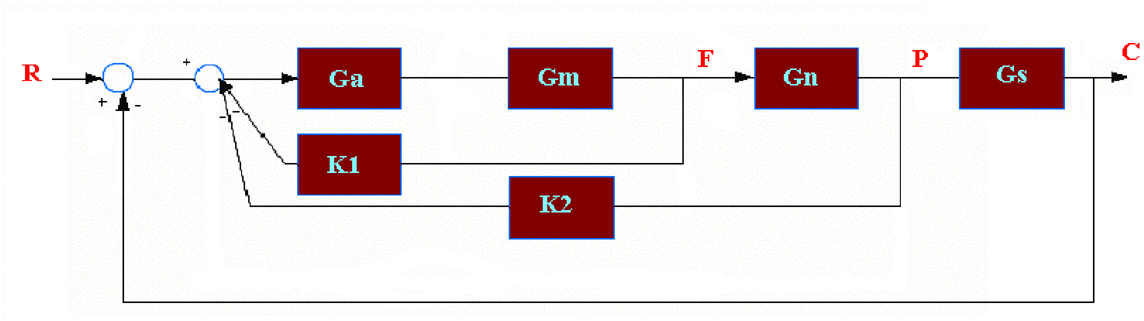
$$\frac{C}{Gp2} = Gp1[D + Gm\left(-Gi\frac{C}{Gp2}\right) + GmGa(R - HC)]$$

$$\left(\frac{1}{Gp2} + \frac{Gp1GmGi}{Gp2} + Gp1GmGa\right)C = Gp1D + Gp1GmGaR$$

PTF = $\frac{C}{R} = \frac{Gp1GmGa}{\left(\frac{1}{Gp2} + \frac{Gp1GmGi}{Gp2} + Gp1GmGa\right)}$

DTF = $\frac{C}{D} = \frac{Gp1}{\left(\frac{1}{Gp1} + \frac{Gp1GmGi}{Gp2} + Gp1GmGa\right)}$

Mixed System # 5



F = GmGa(R – K1F –K2P – C)   (1)

P = GnF   or $F = \frac{P}{Gn}$ (2)

C = GsP   or $C = GsGnF$ or $F = \frac{C}{GsGn}$ or $P = \frac{C}{Gs}$ (3)

Sub in Values into Equation (1)

$$\frac{C}{GsGn} = GmGa\left(R - K1\frac{C}{GsGn} - K2\frac{C}{Gs} - C\right)$$

$$\left(\frac{1}{GsGn} + \frac{K1GmGa}{GsGn} + \frac{K2GmGa}{Gs} + 1\right)C = GmGaR$$

PTF = $\frac{C}{R} = \frac{GmGa}{\left(\frac{1}{GsGn} + \frac{K1GmGa}{GsGn} + \frac{K2GmGa}{Gs} + 1\right)}$

Chapter

8

CONTINUOUS CONTROL

MAE 443

Root Locus

Root Locus

The general form of an equation whose roots can be studied by the root locus method is

$$D(s) + KN(s) = 0$$

Where K is the parameter varied. For now, we take the function D(s) and N(s) to be polynomials in "s" with constant coefficients, and we consider the case K ≥ 0. Another standard form of the problem is obtained by rewriting the previous equation as

$$1 + KP(s) = 0$$

$$\text{Where } P(s) = \frac{N(s)}{D(s)}$$

Plotting Guides for the Primary Root Locus:

Standard Form:

$$1 + KP(s) = 0 \;, \quad K \geq 0$$

$$P(s) = \frac{N(s)}{D(s)}$$

$$N(s) = s^m + b_{m-1}s^{m-1} + \dots\dots\dots + b_1 s + b_0$$

$$D(s) = s^n + a_{n-1}s^{n-1} + \dots\dots\dots + a_1 s + a_0$$

$$m \leq n$$

Basis Terminology:

The "Zeros" of P(s) are the roots of N(s) = 0

The "Poles" of P(s) are the roots of D(s) = 0

Guides to plotting Root Locus

Guide 1:

The Root Locus plot is symmetric about the real axis. This is because complex roots occur in conjugate pairs. Thus, we need deal with upper half-plane of the plot.

Guide 2:

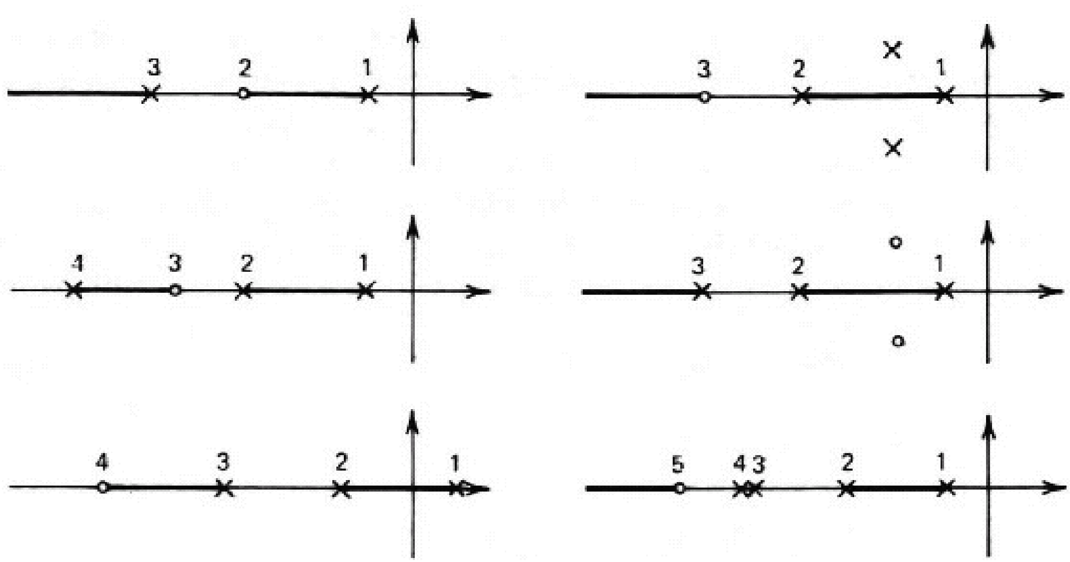
The number of loci equals the number of poles of P(s).

Guide 3:

The loci start at the poles of P(s) with K = 0 and terminate with K = ∞ either at the zeros of P(s) or at infinity.

Guide 4:

The root locus can exist on the real axis only on the left of an odd number of real poles and/or zeros; furthermore, it must exist there. The figure below represents the location of the Root Locus on the real axis for various pole-zero configurations. The locus off the real axis is not shown. X's are Poles and O's are Zero's



Guide 5:

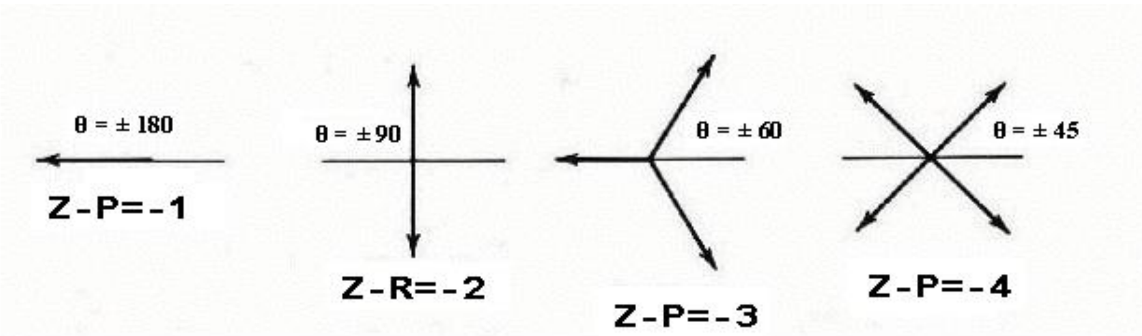
The locations of breakaway and break-in points are found by determining where the parameter K attains a local maximum or minimum on the real axis.

Guide 6:

The loci that do not terminate at a zero approach infinity along asymptotes. The angles that the asymptotes make with the real axis are found from the following equation:

$$\theta = \frac{n180°}{Z-P} \qquad n = \pm 1\ ,\ \pm 3, \ldots\ldots\ldots$$

n is chosen successively as n = +1, -1, +3, -3,….., until enough angles have been found. The following figure shows some common asymptotic angles.



Guide 7:

The asymptotes intersect the real axis at the common point:

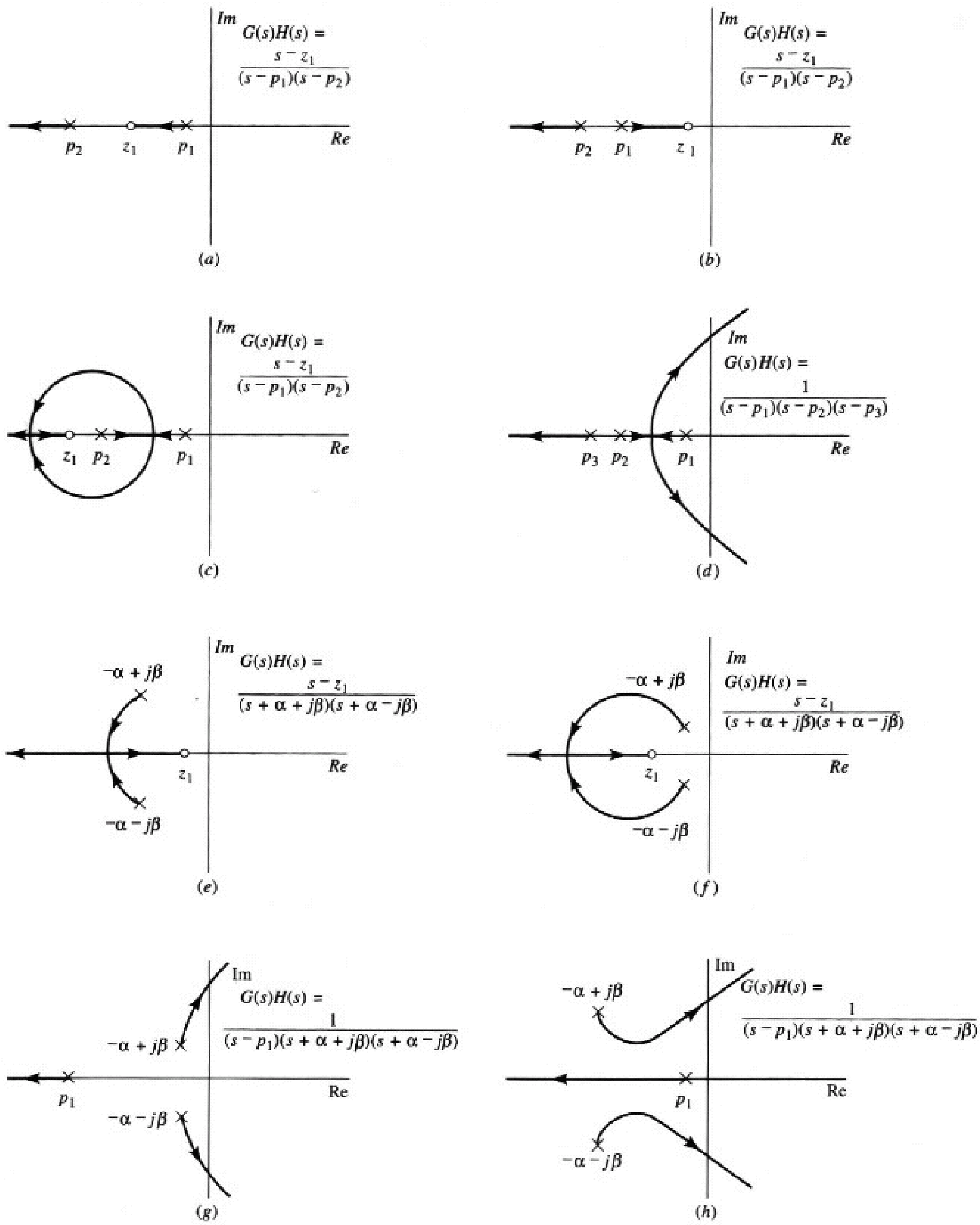
$$\sigma = \frac{\sum s_p - \sum s_z}{P-Z}$$

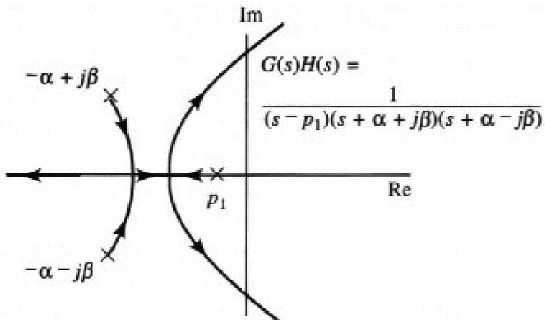
Where $\sum s_p$ and $\sum s_z$ are the algebraic sums of the values of the poles and zeros.

Guide 8:

The points at which the loci cross the imaginary axis and the associated values of K can be found by the Routh-Hurwitz criterion or by substituting s = $i\omega$ into the equation of interest. The frequency ω is the cross over frequency.

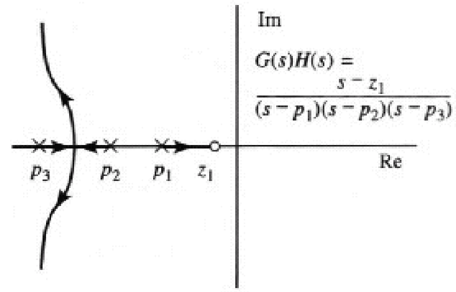
Here are Some Common Root Locus Plots





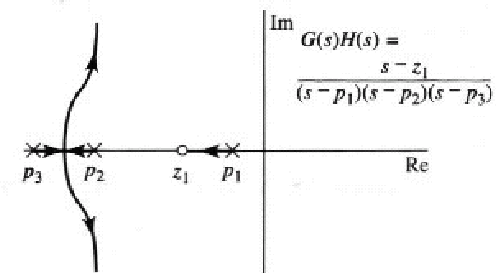
Im
$G(s)H(s) = \frac{1}{(s-p_1)(s+\alpha+j\beta)(s+\alpha-j\beta)}$
$-\alpha + j\beta$
$-\alpha - j\beta$
p_1
Re

(i)



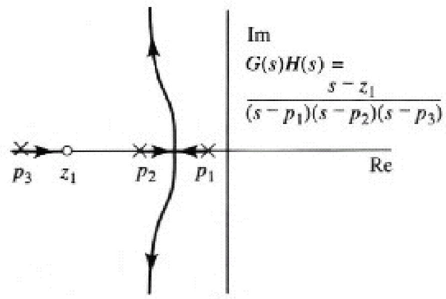
Im
$G(s)H(s) = \frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)}$
p_3 p_2 p_1 z_1
Re

(j)



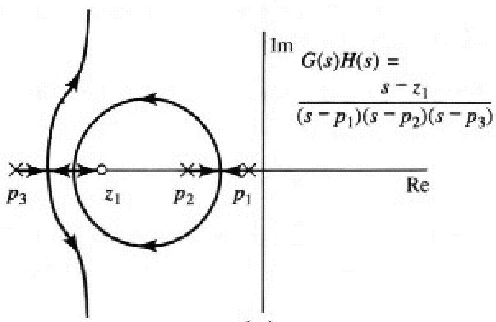
Im
$G(s)H(s) = \frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)}$
p_3 p_2 z_1 p_1
Re

(k)



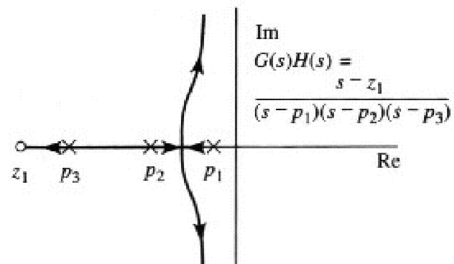
Im
$G(s)H(s) = \frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)}$
p_3 z_1 p_2 p_1
Re

(l)



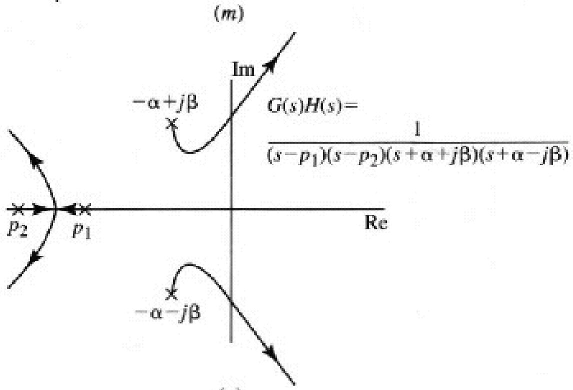
Im
$G(s)H(s) = \frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)}$
p_3 z_1 p_2 p_1
Re

(m)



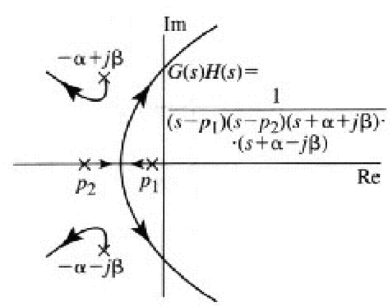
Im
$G(s)H(s) = \frac{s-z_1}{(s-p_1)(s-p_2)(s-p_3)}$
z_1 p_3 p_2 p_1
Re

(n)



Im
$G(s)H(s) = \frac{1}{(s-p_1)(s-p_2)(s+\alpha+j\beta)(s+\alpha-j\beta)}$
$-\alpha+j\beta$
$-\alpha-j\beta$
p_2 p_1
Re

(o)



Im
$G(s)H(s) = \frac{1}{(s-p_1)(s-p_2)(s+\alpha+j\beta)\cdot \cdot(s+\alpha-j\beta)}$
$-\alpha+j\beta$
$-\alpha-j\beta$
p_2 p_1
Re

(p)

EXAMPLES

Example # 1 Draw the Root Locus plot for the range $0 \le K \le \infty$ for the roots of the characteristic polynomial given by $s^3 + 4s^2 + 5s + K = 0$

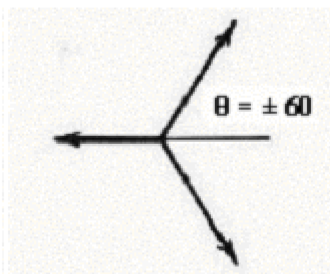
$P = s^3 + 4s^2 + 5s$ $\qquad Q = 1$

$P' = 3s^2 + 8s + 5$ $\qquad Q' = 0$

Poles at $s_1 = 0$ Zeros = None

$s_{2,3} = -2 \pm i$

Asymptotes: 3 Poles – 0 Zeros = 3

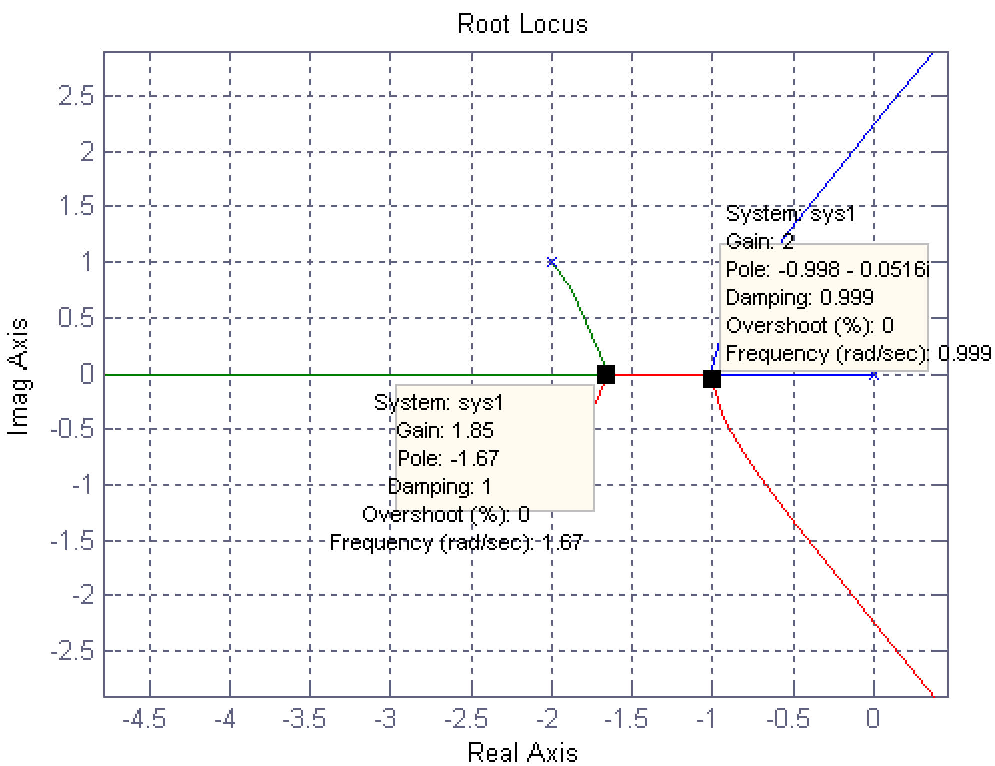


Intersection of Asymptotes:

$$\sigma = \frac{\sum s_p - \sum s_z}{P - Z} = \frac{(0 - 2 - 2) - (0)}{3} = -1.33$$

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 8s + 5)(1) - (s^3 + 4s^2 + 5s)(0) = 0$$
$$3s^2 + 8s + 5 = 0$$
$$s_1 = -1$$
$$s_2 = -1.667$$



QA. Determine the value of K for optimal speed response, and corresponding time constant.

Fastest occurs at when s = -1 $\therefore$ to find corresponding K value substitute s = -1 into the characteristic polynomial $s^3 + 4s^2 + 5s + K = 0$

$\therefore$ K = 2, To find Time Constant: $\tau = \frac{-1}{-s} = \frac{-1}{-1} = 1 \text{ second}$

QB. Same as QA, but with added constraint that the response must be non-oscillatory (i.e., non sinusoidal component), and the corresponding time constant.

The Fastest Non-Oscillatory Response is always at s = -1

$\therefore$ K = 2, To find Time Constant: $\tau = \frac{-1}{-s} = \frac{-1}{-1} = 1 \text{ second}$

QC. Determine the entire range of values of K (if any) for which the response is non-oscillatory.

According to root locus plot, the range for the non-oscillatory occurs when s = -1 and s = -1.67 $\therefore$ Substitute these values into $s^3 + 4s^2 + 5s + K = 0$ and find corresponding K values. $\therefore$ K = 2 and K = 1.85 $\therefore$ 2 > K > 1.85

QD. Find the entire range of values of K (if any) for which the response is stable.

To find stability put: $s^3 + 4s^2 + 5s + K = 0$ into Routh Table:

Original Conditions: K > 0

Routh Table

s^3	1	5
s^2	4	K
s^1	$\frac{20-K}{2}$	0
s^0	K	

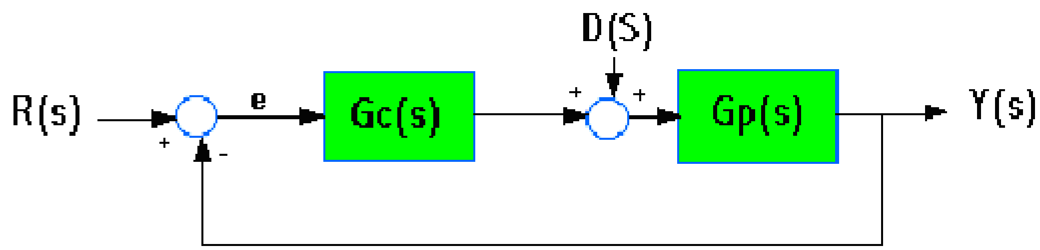
New Conditions from Routh criterion: K < 20

Therefore Final Conditions for Stability are: 20 > K > 0

Matlab Code:

```
>>sys1=tf([1];[1 2 3 6]);  %  tf( Q(s), P(s)
>>rlocus(sys1),grid   % Plots Root Locus with grid
```

Example # 2 Draw the Root Locus plot for the range $0 \le K \le \infty$



Gp = $\dfrac{1}{s^3+2s^2+3s+6}$ and Gc = Kp

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\dfrac{1}{s^3+2s^2+3s+6}\right)(Kp)}{1+(Kp)\left(\dfrac{1}{s^3+2s^2+3s+6}\right)} * \frac{(s^3+2s^2+3s+6)}{(s^3+2s^2+3s+6)}$$

$$= \frac{Kp}{s^3+2s^2+3s+6+Kp}$$

The characteristic polynomial given by $s^3+2s^2+3s+6+Kp=0$

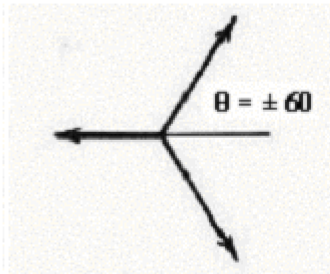
$P = s^3+2s^2+3s+6$ $\qquad Q=1$

$P' = 3s^2+4s+3$ $\qquad Q'=0$

Poles at s_1 = -2 $\qquad$ Zeros = None

$s_{2,3} = 0 \pm i1.731$

Asymptotes: 3 Poles – 0 Zeros = 3

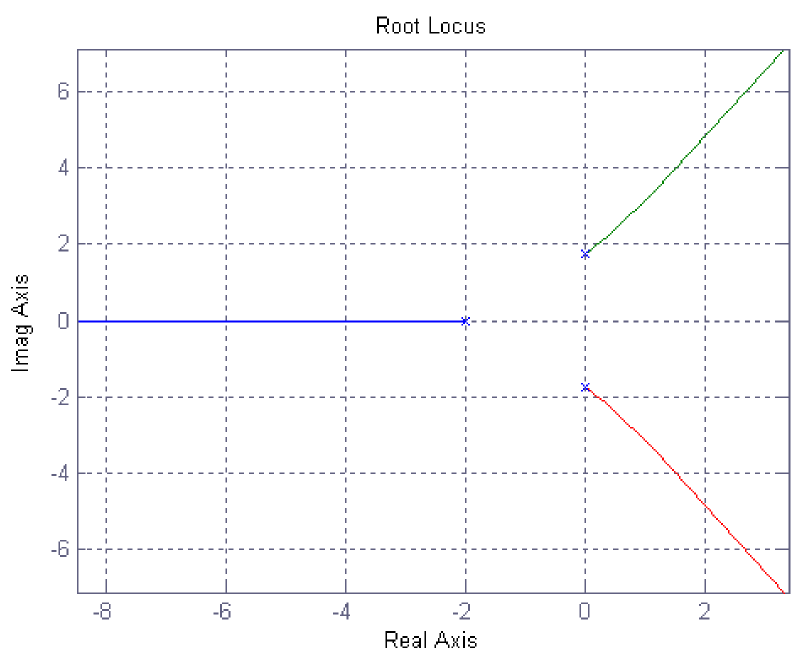


Intersection of Asymptotes:

$$\sigma = \frac{\sum s_p - \sum s_z}{P - Z} = \frac{(-2-0-0) - (0)}{3} = -0.66$$

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 4s + 3)(1) - (s^3 + 2s^2 + 3s + 6)(0) = 0$$
$$3s^2 + 4s + 3 = 0$$
$$s_1 = -0.667$$
$$s_2 = 0.745$$



QA. Find the entire range of values of Kp (if any) for which the response is stable.

$$s^3 + 2s^2 + 3s + (6 + Kp) = 0$$

Original Conditions: Kp > - 6

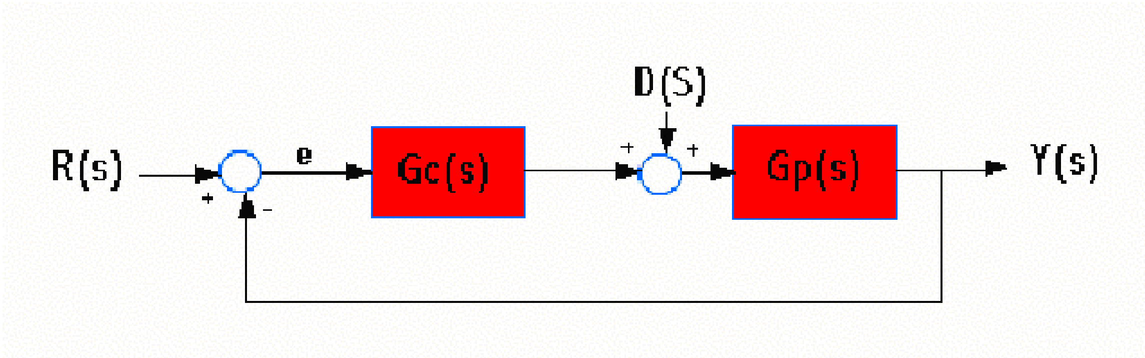
Routh Table

s^3	1	3
s^2	2	6+Kp
s^1	$\frac{(3)(2)-(1)(6+Kp)}{2} = \frac{-Kp}{2}$	0
s^0	6+Kp	

New Conditions from Routh criterion: Kp > 0

Therefore Final Conditions for Stability are: 0 > Kp > - 6

Example # 3 Draw the Root Locus plot for the range $0 \le K \le \infty$



Gp = $\dfrac{1}{s^2+9s+20}$ and Gc = $K\left(1+\dfrac{12}{s}\right)$ (i.e., PI control with Ki = 12 Kp

K is really Kp because Kp = $\dfrac{Ki}{12}$ $\quad \therefore \; Kp+\dfrac{12Kp}{s}=Kp+\dfrac{Ki}{s}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)}=\frac{GpGc}{1+GpGc}=\frac{\left(\dfrac{1}{s^2+9s+20}\right)\left(K\left(1+\dfrac{12}{s}\right)\right)}{1+\left(K\left(1+\dfrac{12}{s}\right)\right)\left(\dfrac{1}{s^2+9s+20}\right)}*\frac{(s^2+9s+20)*s}{(s^2+9s+20)*s}$$

$$=\frac{K(s+12)}{s^3+9s^2+20s+(K(s+12))}$$

$$=\frac{K(s+12)}{s^3+9s^2+(20+K)s+12K}$$

$$P = s^2 + 9s + 20$$
$$P = s^3 + 9s^2 + 20s$$
$$P' = 3s^2 + 18s + 20$$

$$Q = \left(1 + \frac{12}{s}\right)$$
$$Q = s + 12$$
$$Q' = 1$$

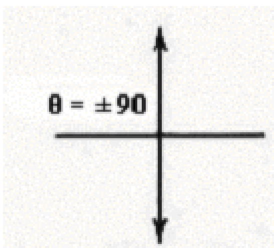
Poles at s_1 = 0

Zeros = - 12

s_2 = - 4

s_3 = - 5

Asymptotes: 3 Poles – 1 Zeros = 2

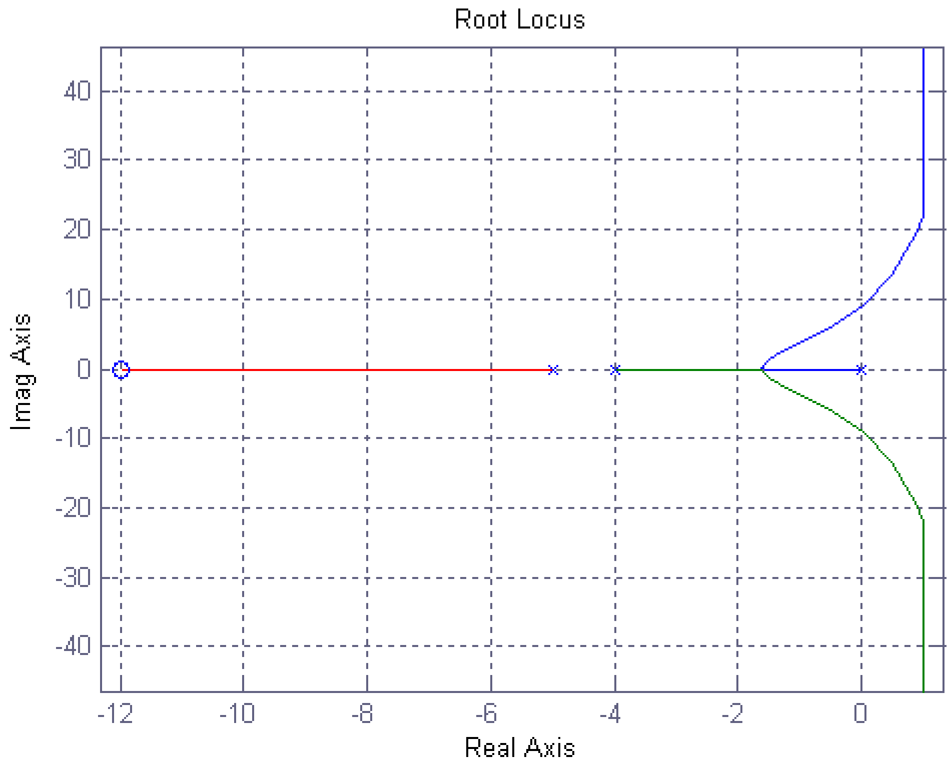


Intersection of Asymptotes:

$$\sigma = \frac{\sum s_p - \sum s_z}{P - Z} = \frac{(0 - 5 - 4) - (-12)}{2} = +1.5$$

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 18s + 20)(12) - (s^3 + 9s^2 + 20s)(1) = 0$$
$$2s^3 + 45s^2 + 216s + 240 = 0$$
$$s_1 = -16.433$$
$$s_2 = -4.5439$$
$$s_3 = -1.616$$



QA. Determine the value of K for optimal speed response, and corresponding time constant.

Fastest occurs at when s = -1.66 $\therefore$ to find corresponding K value substitute s = -1 into the characteristic polynomial $s^3 + 9s^2 + (20 + K)s + 12K = 0$

$\therefore$ K = 1.25 , To find Time Constant: $\tau = \dfrac{-1}{-s} = \dfrac{-1}{-1.66} = 0.618$ second

QB. Same as QA, but with added constraint that the response must be non-oscillatory (i.e., non sinusoidal component), and the corresponding time constant.

The Fastest Non-Oscillatory Response is always at s = -1.66

$\therefore$ K = 1.25, To find Time Constant: $\tau = \dfrac{-1}{-s} = \dfrac{-1}{-1.66} = 0.618$ second

QC. Determine the entire range of values of K (if any) for which the response is non-oscillatory.

According to root locus plot, the range for the non-oscillatory occurs when s = -1.66 and s = 1.5 $\therefore$ Substitute these values into $s^3 + 9s^2 + (20 + K)s + 12K = 0$ and find corresponding K values. $\therefore$ K = 1.25 and K = 60 $\therefore$ $1.25 \geq K \geq 0$ and K < 60

QD. Find the entire range of values of K (if any) for which the response is stable.

$$s^3 + 9s^2 + (20 + K)s + 12K = 0$$

Original Conditions: K > - 20 and K > 0

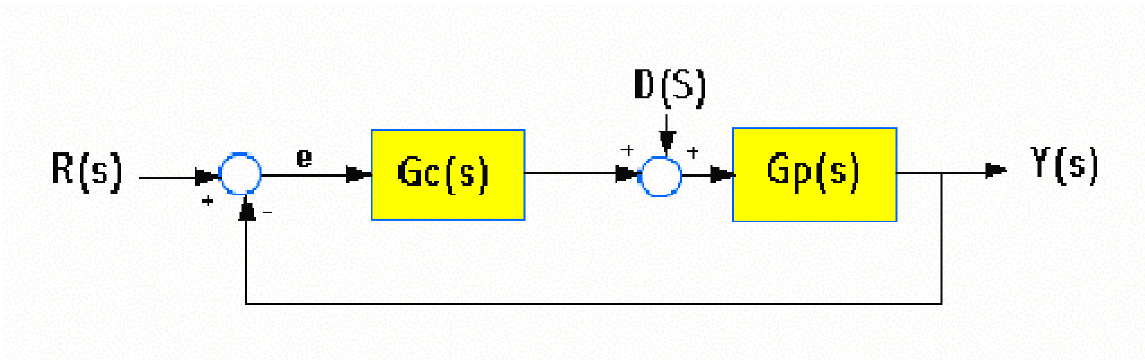
Routh Table

s^3	1	(20+K)
s^2	9	12K
s^1	$\frac{(9)(20+K)-(1)(12K)}{9} = \frac{180-3K}{9}$	0
s^0	12K	

New Conditions from Routh criterion: K > 0 and K < 60

Therefore Final Conditions for Stability are: 0 < K < 60

Example # 4 Draw the Root Locus plot for the range $0 \le K \le \infty$



Gp = $\dfrac{1}{s^3+9s^2+20s+12}$ and Gc = K(s + 2.1) (i.e., PD control with Kp = 2.1 Kd

K is really Kd because Kd = $\dfrac{Kp}{2.1}$ $\therefore\ Kds + 2.1Kd = Kds + Kp$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\dfrac{1}{s^3+9s^2+20s+12}\right)(K(s+2.1))}{1+(K(s+2.1))\left(\dfrac{1}{s^3+9s^2+20s+12}\right)} * \frac{(s^3+9s^2+20s+12)}{(s^3+9s^2+20s+12)}$$

$$= \frac{K(s+2.1)}{s^3+9s^2+20s+12+(K(s+2.1))}$$

$$= \frac{K(s+2.1)}{s^3+9s^2+(20+K)s+(12+2.1K)}$$

$P = s^3 + 9s^2 + 20s + 12$

$P' = 3s^2 + 18s + 20$

$Q = s + 2.1$

$Q' = 1$

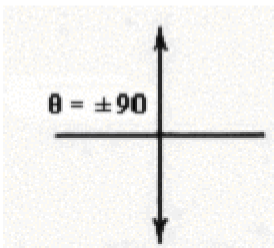
Poles at s_1 = - 1

s_2 = - 2

s_3 = - 6

Zeros = - 2.1

Asymptotes: 3 Poles – 1 Zeros = 2

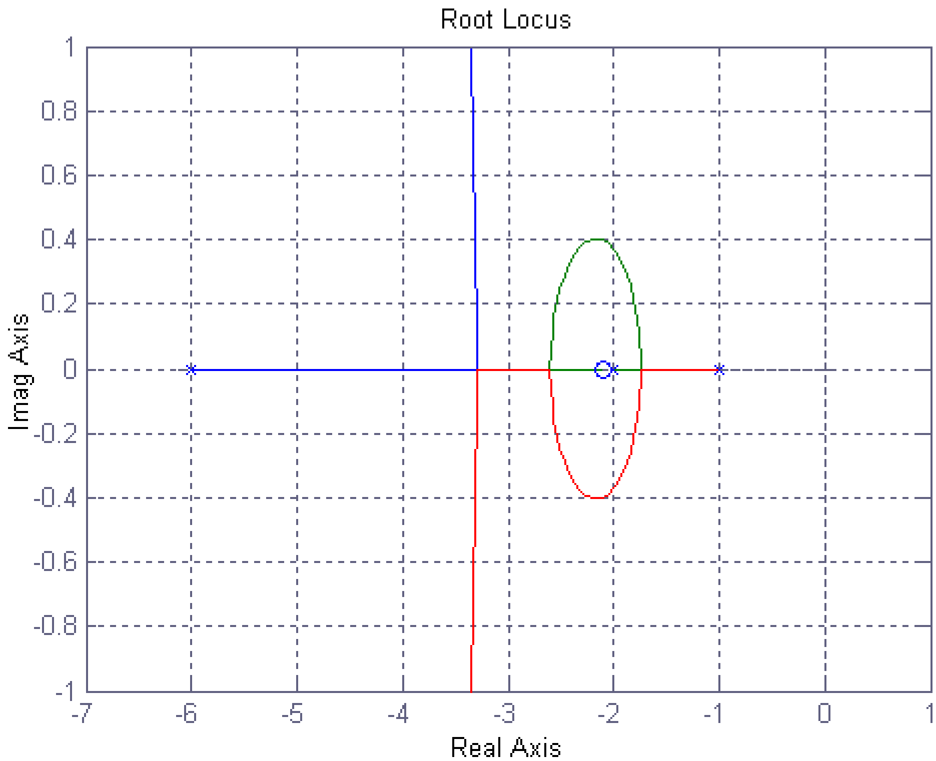


Intersection of Asymptotes:

$$\sigma = \frac{\sum s_p - \sum s_z}{P - Z} = \frac{(-6-2-1)-(-2.1)}{2} = -3.45$$

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 18s + 20)(s + 2.1) - (s^3 + 9s^2 + 20s + 12)(1) = 0$$
$$2s^3 + 153s^2 + 37.8s + 30 = 0$$
$$s_1 = -1.745$$
$$s_2 = -2.603$$
$$s_3 = -3.30$$



QA. Determine the value of K for optimal speed response, and corresponding time constant.

Fastest occurs at first break out point, but at that point it takes you over to s = - 3.3

$\therefore$ to find corresponding K value substitute s = - 3.3 into the characteristic polynomial

$$s^3 + 9s^2 + (20 + K)s + (12 + 1.2K) = 0$$

$\therefore$ K = 6.53 , To find Time Constant: $\tau = \frac{-1}{-s} = \frac{-1}{-3.3} = 0.384$ second

QB. Same as QA, but with added constraint that the response must be non-oscillatory (i.e., non sinusoidal component), and the corresponding time constant.

The Fastest Non-Oscillatory Response is the same as QA: at s = - 3.3

$\therefore$ K = 6.53 , To find Time Constant: $\tau = \frac{-1}{-s} = \frac{-1}{-3.3} = 0.384$ second

QC. Determine the entire range of values of K (if any) for which the response is non-oscillatory.

According to root locus plot, the range for the non-oscillatory occurs when s = - 1.745 and s = - 2.603 and s = - 3.3 $\therefore$ Sub values in: $s^3 + 9s^2 + (20 + K)s + (12 + 1.2K) = 0$

and find corresponding K values. $\therefore$ K = 2.27, K = 6.528, and K = 6.73

$\therefore$ $2.51 \geq K \geq 0$ and $6.73 \geq K \geq 6.528$

QD. Find the entire range of values of K (if any) for which the response is stable.

$$s^3 + 9s^2 + (20 + K)s + (12 + 1.2K) = 0$$

Original Conditions: K > - 20 and K > - 5.714

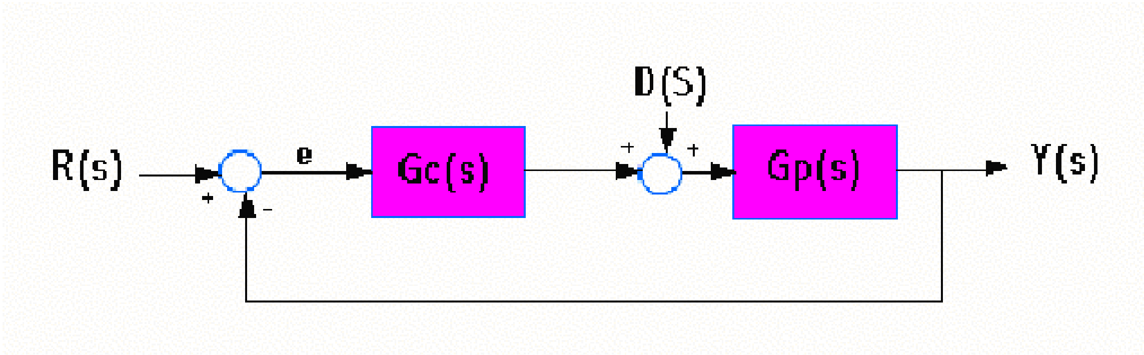
Routh Table

s^3	1	(20+K)
s^2	9	(12+2.1K)
s^1	$\frac{(9)(20+K)-(1)(12+2.1K)}{9} = \frac{6.9K+168}{9}$	0
s^0	(12+2.1K)	

New Conditions from Routh criterion: K > - 5.714 and K > - 24.347

Therefore Final Conditions for Stability are: K > - 5.714

Example # 5 Draw the Root Locus plot for the range $0 \le K \le \infty$



Gp = $\dfrac{1}{s^2+2s+10}$ and Gc = $K\left(s+4+\dfrac{5}{s}\right)$

(i.e., PID control with Kp = 4 Kd and Ki = 5 Kd

K is really Kd because Kd = $\dfrac{Kp}{4}$ and Kd = $\dfrac{Ki}{5}$ $\therefore$ $Kds + 4Kd + \dfrac{5Kd}{s} = Kds + Kp + \dfrac{Ki}{s}$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\dfrac{1}{s^2+2s+10}\right)\left(K\left(s+4+\dfrac{5}{s}\right)\right)}{1+\left(K\left(s+4+\dfrac{5}{s}\right)\right)\left(\dfrac{1}{s^2+2s+10}\right)} * \frac{(s^2+2s+10)*s}{(s^2+2s+10)*s}$$

$$= \frac{K(s^2+4s+5)}{s^3+2s^2+10s+(K(s^2+4s+5))}$$

$$= \frac{K(s^2+4s+5)}{s^3+(2+K)s^2+(10+4K)s+5K}$$

$$P = s^2 + 2s + 10$$
$$P = s^3 + 2s^2 + 10s$$
$$P' = 3s^2 + 4s + 10$$

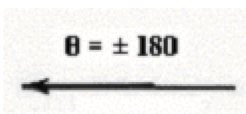
$$Q = \left(s + 4 + \frac{5}{s} \right)$$
$$Q = s^2 + 4s + 5$$
$$Q' = 2s + 4$$

Poles at $s_1 = 0$

$s_{2,3} = -1 \pm 3i$

Zeros at $s_{1,2} = -2 \pm i$

Asymptotes: 3 Poles – 2 Zeros = 1

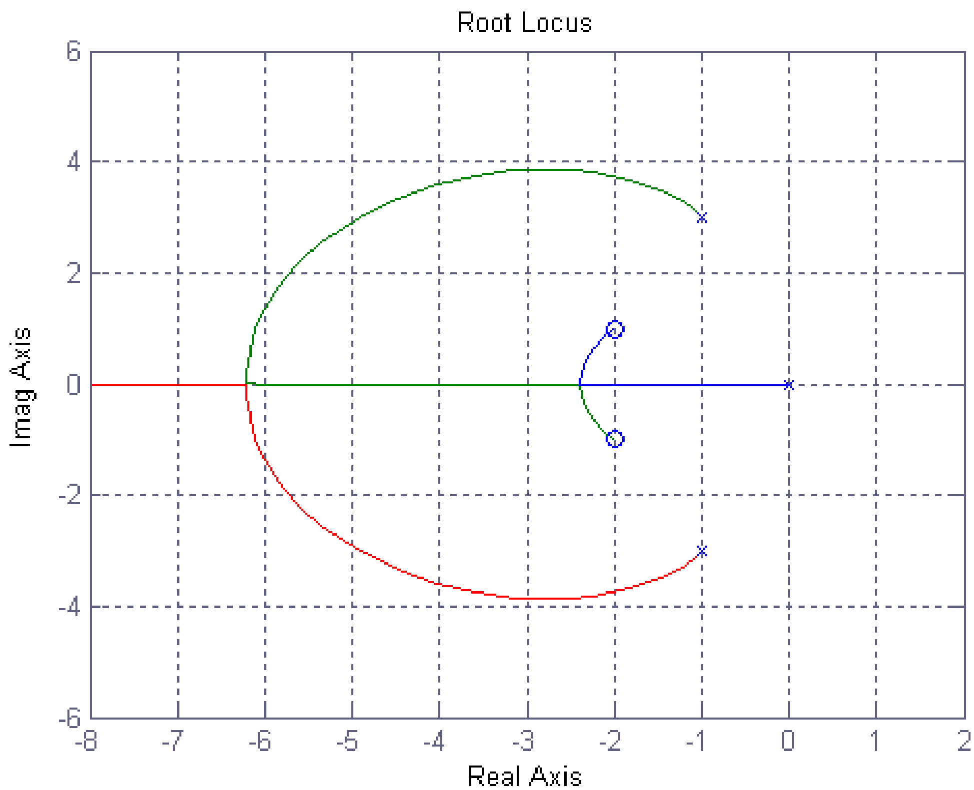


Intersection of Asymptotes:

$$\sigma = \frac{\sum s_p - \sum s_z}{P - Z}$$ Since there is only 1 asymptote: There is no intersection

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 4s + 10)(s^2 + 4s + 5) - (s^3 + 2s^2 + 10s)(2s + 4) = 0$$
$$s^4 + 8s^3 + 13s^2 + 20s + 150 = 0$$
$$s_1 = -6.218$$
$$s_2 = -2.3868$$
$$s_{3,4} = .308 \pm 1.08i$$


Root Locus
Imag Axis
Real Axis
6
4
2
0
-2
-4
-6
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2

Example # 6 Design a simple feedback control system by using a "P-I-D" such that the design goals are achieved:

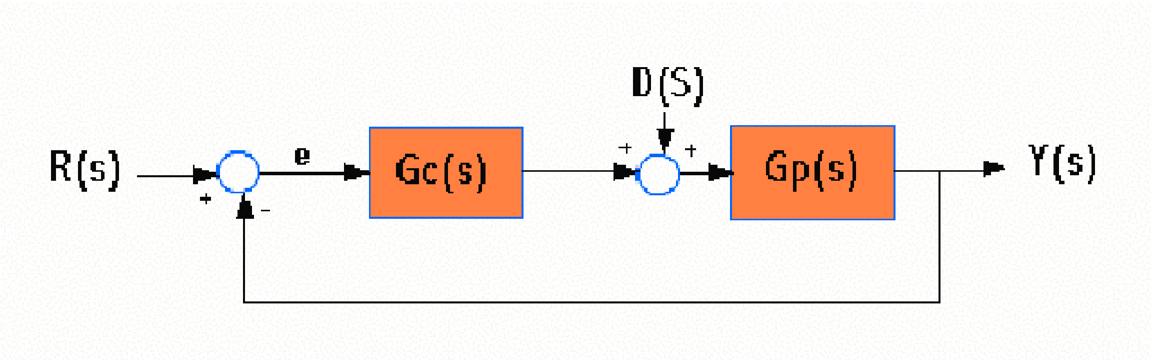
A) The system is stable

B) The time constant is less than 0.5

C) The steady-state response to a unit step command input is 1. (Perfect Accuracy!)

D) The steady-state response to a unit disturbance input is 0. (Perfect Robustness)

E) Draw the Root Locus plot for the range $0 \le K \le \infty$



Gp = $\dfrac{1}{s^2+2s}$ and Gc = $\left(Kds + Kp + \dfrac{Ki}{s}\right)$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\dfrac{1}{s^2+2s}\right)\left(Kds+Kp+\dfrac{Ki}{s}\right)}{1+\left(Kds+Kp+\dfrac{Ki}{s}\right)\left(\dfrac{1}{s^2+2s}\right)} * \frac{(s^2+2s)*s}{(s^2+2s)*s}$$

$$= \frac{Kds^2+Kps+Ki}{s^3+2s^2+Kds^2+Kps+Ki}$$

$$= \frac{Kds^2+Kps+Ki}{s^3+(2+Kd)s^2+Kps+Ki}$$

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s^2+2s}\right)}{1+\left(Kds+Kp+\frac{Ki}{s}\right)\left(\frac{1}{s^2+2s}\right)} * \frac{(s^2+2s)*s}{(s^2+2s)*s}$$

$$= \frac{s}{s^3+(2+Kd)s^2+Kps+Ki}$$

iii) Accuracy: $\lim_{s\to 0}(PTF) = \frac{Kds^2+Kps+Ki}{s^3+(2+Kd)s^2+Kps+Ki} = \frac{Ki}{Ki} = 1$

iv) Robustness: $\lim_{s\to 0}(DTF) = \frac{s}{s^2+(2+Kd)s^2+Kps+Ki} = 0$

i) Stability: Do Routh Table

CE: $s^3+(2+Kd)s^2+Kps+Ki$

Original Conditions: (1) Kd > - 2 , (2) Kp > 0 , (3) Ki > 0

Routh Table

s^3	1	Kp
s^2	(2+Kd)	Ki
s^1	$\frac{(2+Kd)(Kp)-(1)(Ki)}{(2+Kd)}$	0
s^0	Ki	

New Conditions from Routh criterion: (4) $(2+Kd)(Kp)-(1)(Ki)$ > 0

These are the 4 conditions for stability!

ii) Now create with time constant less than 0.5 $\therefore$ sub $\left(s-\frac{1}{\tau}\right)=\left(s-\frac{1}{0.5}\right)$ into CE

$$(s-2)^3+(2+Kd)(s-2)^2+Kp(s-2)+Ki$$
$$s^3+Kds^2-4s^2-4Kds+Kps+4s+4Kd-2Kp+Ki$$
$$s^3+(Kd-4)s^2+(4-4Kd+Kp)s+(4Kd-2Kp+Ki)$$

New Conditions:

(1) Kd > 4

(2) 4 – 4Kd + Kp > 0

(3) 4Kd – 2Kp + Ki > 0

Routh Table

s^3	1	(4-4Kd+Kp)
s^2	(Kd-4)	(4Kd-2Kp+Ki)
s^1	$\frac{(Kd-4)(4-4Kd+Kp)-(1)(4Kd-2Kp+Ki)}{(Kd-4)}$	0
s^0	(4Kd+Ki-2Kp)	

New Conditions from Routh criterion: (4) $16Kd-4Kd^2+KdKp-2Kp-Ki-16>0$

Now we want to Find Values to satisfy all 4 equations

LET Kd = 5

(2) Kp > 16

(3) -2Kp + Ki > -20

(4) 3Kp – Ki > 36

LET Kp = 18

(3) Ki > 16

(4) Ki < 18

Let Ki = 17

$\therefore$ Kd = 5, Kp = 18, and Ki = 17 will satisfy $\tau < 0.5$ seconds

v) Root Locus Plot

Gp = $\dfrac{1}{s^2 + 2s}$ and Gc = K$\left(5s + 18 + \dfrac{17}{s}\right)$

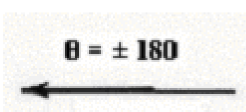
$$P = s^2 + 2s \qquad\qquad Q = \left(5s + 18 + \frac{17}{s}\right)$$
$$P = s^3 + 2s^2 \qquad\qquad Q = 5s^2 + 18s + 17$$
$$P' = 3s^2 + 4s \qquad\qquad Q' = 10s + 18$$

Poles at $s_1 = 0$ Zeros at $s_{1,2} = -1.8 \pm 0.4i$

$s_2 = 0$

$s_3 = -2$

Asymptotes: 3 Poles – 2 Zeros = 1

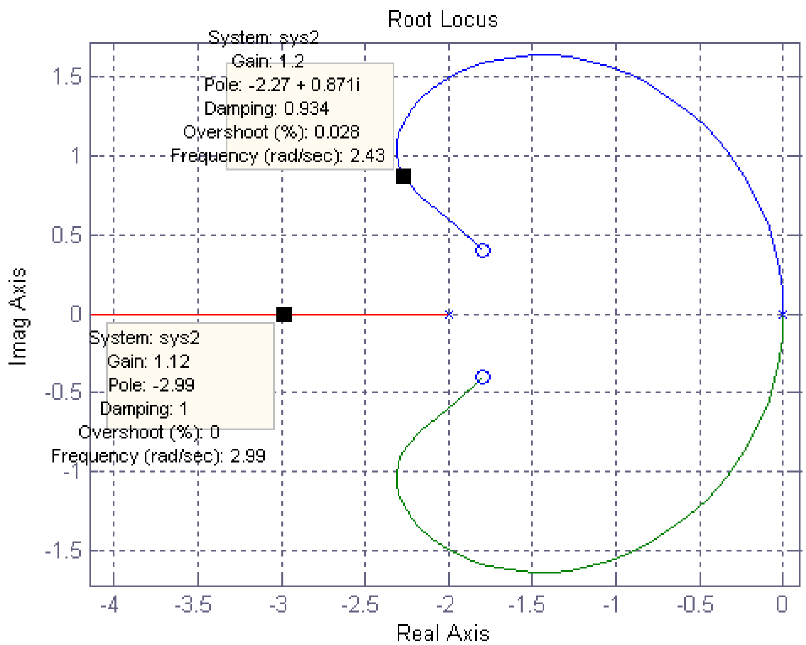


Intersection of Asymptotes:

$\sigma = \dfrac{\sum s_p - \sum s_z}{P - Z}$ Since there is only 1 asymptote: There is no intersection

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 4s)(5s^2 + 18s + 17) - (s^3 + 2s^2)(10s + 18) = 0$$
$$5s^4 + 36s^3 + 87s^2 + 68s = 0$$
$$s_1 = 0$$
$$s_2 = -1.65$$
$$s_{3,4} = -2.77 \pm 0.76i$$



PTF = $\dfrac{Y(s)}{D(s)} = \dfrac{GpGc}{1+GpGc} = \dfrac{\left(\dfrac{1}{s^2+2s}\right)\left(K\left(5s+18+\dfrac{17}{s}\right)\right)}{1+\left(K\left(5s+18+\dfrac{17}{s}\right)\right)\left(\dfrac{1}{s^2+2s}\right)} * \dfrac{(s^2+2s)*s}{(s^2+2s)*s}$

PTF = $\dfrac{K(5s^2+18s+17)}{s^3+2s^2+K(5s^2+18s+17)}$

The fastest system occurs roughly about s = -2.31 $\therefore$ $\tau = \dfrac{-1}{-2.31} = .423$ seconds

Sub s = -2.31 into $s^3 + 2s^2 + K(5s^2 + 18s + 17)$ and K = 1.2

Example # 7 Design a simple feedback control system by using a "P-I-D" such that the design goals are achieved:

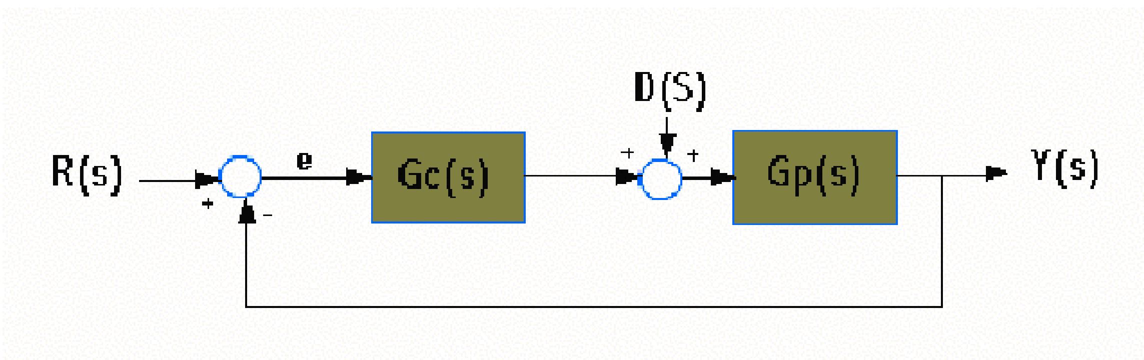
A) The system is stable

B) The time constant is less than 0.5

C) The steady-state response to a unit step command input is 1. (Perfect Accuracy!)

D) The steady-state response to a unit disturbance input is 0. (Perfect Robustness)

E) Draw the Root Locus plot for the range $0 \le K \le \infty$



Gp = $\dfrac{1}{s^2+2s+1}$ and Gc = $\left(Kds + Kp + \dfrac{Ki}{s}\right)$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\dfrac{1}{s^2+2s+1}\right)\left(Kds+Kp+\dfrac{Ki}{s}\right)}{1+\left(Kds+Kp+\dfrac{Ki}{s}\right)\left(\dfrac{1}{s^2+2s+1}\right)} * \frac{(s^2+2s+1)*s}{(s^2+2s+1)*s}$$

$$= \frac{Kds^2+Kps+Ki}{s^3+2s^2+s+Kds^2+Kps+Ki}$$

$$= \frac{Kds^2+Kps+Ki}{s^3+(2+Kd)s^2+(1+Kp)s+Ki}$$

$$\text{DTF} = \frac{Y(s)}{R(s)} = \frac{Gp}{1+GpGc} = \frac{\left(\frac{1}{s^2+2s+1}\right)}{1+\left(Kds+Kp+\frac{Ki}{s}\right)\left(\frac{1}{s^2+2s+1}\right)} * \frac{(s^2+2s+1)*s}{(s^2+2s+1)*s}$$

$$= \frac{s}{s^3+(2+Kd)s^2+(1+Kp)s+Ki}$$

iii) Accuracy: $\lim_{s\to 0}(PTF) = \frac{Kds^2+Kps+Ki}{s^3+(2+Kd)s^2+(1+Kp)s+Ki} = \frac{Ki}{Ki} = 1$

iv) Robustness: $\lim_{s\to 0}(DTF) = \frac{s}{s^2+(2+Kd)s^2+(1+Kp)s+Ki} = 0$

i) Stability: Do Routh Table

CE: $s^3+(2+Kd)s^2+(1+Kp)s+Ki$

Original Conditions: (1) Kd > - 2 , (2) Kp > - 1 , (3) Ki > 0

Routh Table

s^3	1	(1+Kp)
s^2	(2+Kd)	Ki
s^1	$\frac{(2+Kd)(1+Kp)-(1)(Ki)}{(2+Kd)}$	0
s^0	Ki	

New Conditions from Routh criterion: (4) $(2+Kd)(1+Kp)-(1)(Ki) > 0$

These are the 4 conditions for stability!

ii) Now create with time constant less than 0.5 $\therefore$ sub $\left(s-\frac{1}{\tau}\right)=\left(s-\frac{1}{0.5}\right)$ into CE

$$(s-2)^3+(2+Kd)(s-2)^2+(1+Kp)(s-2)+Ki$$
$$s^3+Kds^2-4s^2-4Kds+Kps+5s+4Kd-2Kp-2+Ki$$
$$s^3+(Kd-4)s^2+(Kp-4Kd+5)s+(4Kd-2Kp+Ki-2)$$

New Conditions:

(1) Kd > 4

(2) 5 - 4Kd + Kp > 0

(3) 4Kd – 2Kp + Ki - 2 > 0

Routh Table

s^3	1	(Kp-4Kd+5)
s^2	(Kd-4)	(4Kd-2Kp+Ki-2)
s^1	$\frac{(Kd-4)(Kp-4Kd+5)-(1)(4Kd-2Kp+Ki-2)}{(Kd-4)}$	0
s^0	(4Kd+Ki-2Kp-2)	

New Conditions from Routh criterion: (4) $17Kd-4Kd^2+KdKp-2Kp-Ki-18>0$

Now we want to Find Values to satisfy all 4 equations

LET Kd = 5

(2) Kp > 15

(3) -2Kp + Ki > -18

(4) 3Kp – Ki > 33

LET Kp = 17

(3) Ki > 16

(4) Ki < 18

Let Ki = 17

$\therefore$ Kd = 5, Kp = 17, and Ki = 17 will satisfy $\tau < 0.5$ seconds

v) Root Locus Plot

Gp = $\dfrac{1}{s^2+2s+1}$ and Gc = K$\left(5s+17+\dfrac{17}{s}\right)$

$$P = s^2+2s+1$$
$$P = s^3+2s^2+s$$
$$P' = 3s^2+4s+1$$

$$Q = \left(5s+17+\frac{17}{s}\right)$$
$$Q = 5s^2+17s+17$$
$$Q' = 10s+17$$

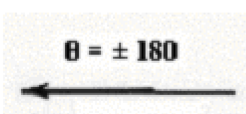
Poles at s_1 = 0

s_2 = - 1

s_3 = - 1

Zeros at $s_{1,2}$ = - 1.7 ± 0.714i

Asymptotes: 3 Poles – 2 Zeros = 1

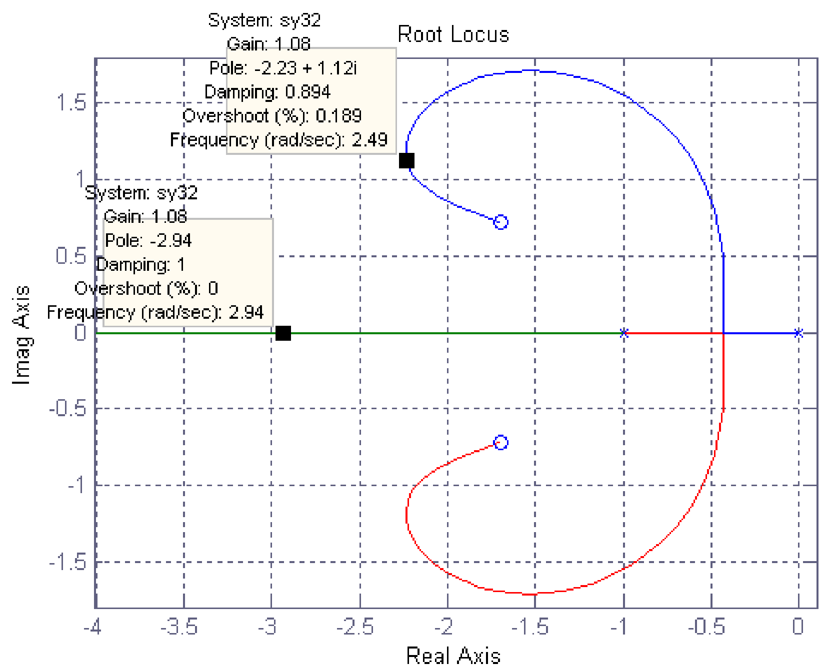


Intersection of Asymptotes:

$\sigma = \dfrac{\sum s_p - \sum s_z}{P-Z}$ Since there is only 1 asymptote: There is no intersection

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 4s + 1)(5s^2 + 17s + 17) - (s^3 + 2s^2 + s)(10s + 17) = 0$$
$$5s^4 + 34s^3 + 80s^2 + 68s + 17 = 0$$
$$s_1 = -1$$
$$s_2 = -0.432$$
$$s_{3,4} = -2.68 \pm 0.824i$$



$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1 + GpGc} = \frac{\left(\frac{1}{s^2 + 2s + 1}\right)\left(K\left(5s + 17 + \frac{17}{s}\right)\right)}{1 + \left(K\left(5s + 18 + \frac{17}{s}\right)\right)\left(\frac{1}{s^2 + 2s + 1}\right)} * \frac{(s^2 + 2s + 1) * s}{(s^2 + 2s + 1) * s}$$

$$\text{PTF} = \frac{K(5s^2 + 17s + 17)}{s^3 + 2s^2 + s + K(5s^2 + 17s + 17)}$$

The fastest system occurs roughly about s = -2.24 $\therefore \ \tau = \frac{-1}{-2.24} = .446$ seconds

Sub s = -2.24 into $s^3 + 2s^2 + s + K(5s^2 + 18s + 17)$ and K = 1.12

Example # 8 Design a simple feedback control system by using a "P-I-D" such that the design goals are achieved:

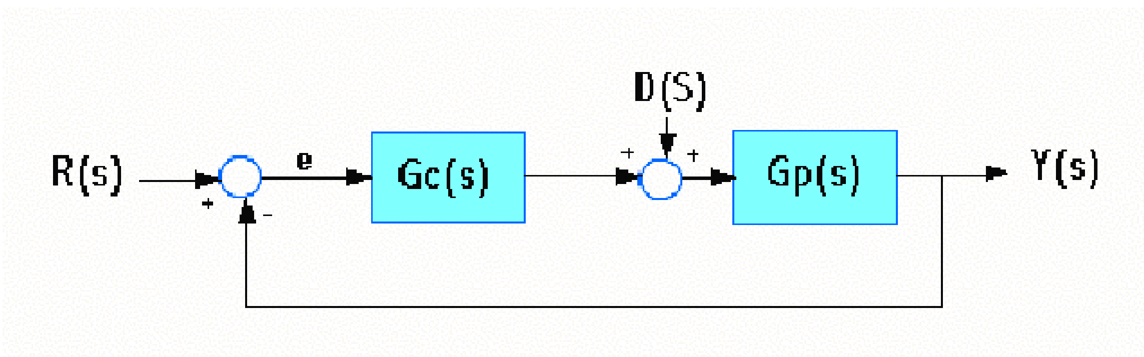
A) *The system is stable*

B) *The time constant is less than 0.25*

C) *The steady-state response to a unit step command input is 1. (Perfect Accuracy!)*

D) *The steady-state response to a unit disturbance input is 0. (Perfect Robustness)*

E) *Draw the Root Locus plot for the range* $0 \le K \le \infty$



$$Gp = \frac{1}{s^2+2s+5} \quad \text{and } Gc = \left(Kds + Kp + \frac{Ki}{s}\right)$$

$\therefore$ Y=Gp[D + Gc(R – Y)]

(1 + GpGc)Y =GpD + GpGcR

The Primary Transfer Function

$$\text{PTF} = \frac{Y(s)}{D(s)} = \frac{GpGc}{1+GpGc} = \frac{\left(\frac{1}{s^2+2s+5}\right)\left(Kds+Kp+\frac{Ki}{s}\right)}{1+\left(Kds+Kp+\frac{Ki}{s}\right)\left(\frac{1}{s^2+2s+5}\right)} * \frac{(s^2+2s+5)*s}{(s^2+2s+5)*s}$$

$$= \frac{Kds^2+Kps+Ki}{s^3+2s^2+5s+Kds^2+Kps+Ki}$$

$$= \frac{Kds^2+Kps+Ki}{s^3+(2+Kd)s^2+(5+Kp)s+Ki}$$

DTF = $\dfrac{Y(s)}{R(s)} = \dfrac{Gp}{1+GpGc} = \dfrac{\left(\dfrac{1}{s^2+2s+5}\right)}{1+\left(Kds+Kp+\dfrac{Ki}{s}\right)\left(\dfrac{1}{s^2+2s+5}\right)} * \dfrac{(s^2+2s+5)*s}{(s^2+2s+5)*s}$

$$= \frac{s}{s^3+(2+Kd)s^2+(5+Kp)s+Ki}$$

iii) Accuracy: $\lim_{s\to 0}(PTF) = \dfrac{Kds^2+Kps+Ki}{s^3+(2+Kd)s^2+(5+Kp)s+Ki} = \dfrac{Ki}{Ki} = 1$

iv) Robustness: $\lim_{s\to 0}(DTF) = \dfrac{s}{s^2+(2+Kd)s^2+(5+Kp)s+Ki} = 0$

i) Stability: Do Routh Table

CE: $s^3+(2+Kd)s^2+(5+Kp)s+Ki$

Original Conditions: (1) Kd > - 2 , (2) Kp > - 5 , (3) Ki > 0

Routh Table

s^3	1	(5+Kp)
s^2	(2+Kd)	Ki
s^1	$\frac{(2+Kd)(5+Kp)-(1)(Ki)}{(2+Kd)}$	0
s^0	Ki	

New Conditions from Routh criterion: (4) $(2+Kd)(5+Kp)-(1)(Ki)$ > 0

These are the 4 conditions for stability!

ii) Now create with time constant less than 0.25 $\therefore$ sub $\left(s-\frac{1}{\tau}\right)=\left(s-\frac{1}{0.25}\right)$ into CE

$$(s-4)^3+(2+Kd)(s-4)^2+(5+Kp)(s-4)+Ki$$
$$s^3+Kds^2-10s^2-8Kds+Kps+37s+16Kd-4Kp-52+Ki$$
$$s^3+(Kd-10)s^2+(Kp-8Kd+37)s+(16Kd-4Kp+Ki-52)$$

New Conditions:

(1) Kd > 10

(2) 37 - 8Kd + Kp > 0

(3) 16Kd – 4Kp + Ki - 52 > 0

Routh Table

s^3	1	(Kp-8Kd+37)
s^2	(Kd-10)	(16Kd-4Kp+Ki-52)
s^1	$\frac{(Kd-10)(Kp-8Kd+37)-(1)(16Kd-4Kp+Ki-52)}{(Kd-10)}$	0
s^0	(16Kd+Ki-4Kp-52)	

New Conditions from Routh criterion: (4) $101Kd-8Kd^2+KdKp-6Kp-Ki-318>0$

Now we want to Find Values to satisfy all 4 equations

LET Kd = 12

(2) Kp > 59

(3) -4Kp + Ki > -140

(4) 6Kp – Ki > 258

LET Kp = 60

(3) Ki > 100

(4) Ki < 102

Let Ki = 101

$\therefore$ Kd = 5, Kp = 60, and Ki = 101 will satisfy $\tau < 0.25$ seconds

v) Root Locus Plot

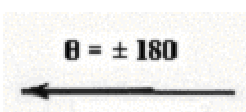
Gp = $\dfrac{1}{s^2+2s+5}$ and Gc = $K\left(12s+60+\dfrac{101}{s}\right)$

$$P = s^2+2s+5 \qquad\qquad Q=\left(12s+60+\frac{101}{s}\right)$$
$$P = s^3+2s^2+5s \qquad\qquad Q=12s^2+60s+101$$
$$P' = 3s^2+4s+5 \qquad\qquad Q'=24s+60$$

Poles at s_1 = 0 Zeros at $s_{1,2}$ = - 1.5 ± 1.472i

$s_{2,3}$ = - 1 ± 2i

Asymptotes: 3 Poles – 2 Zeros = 1

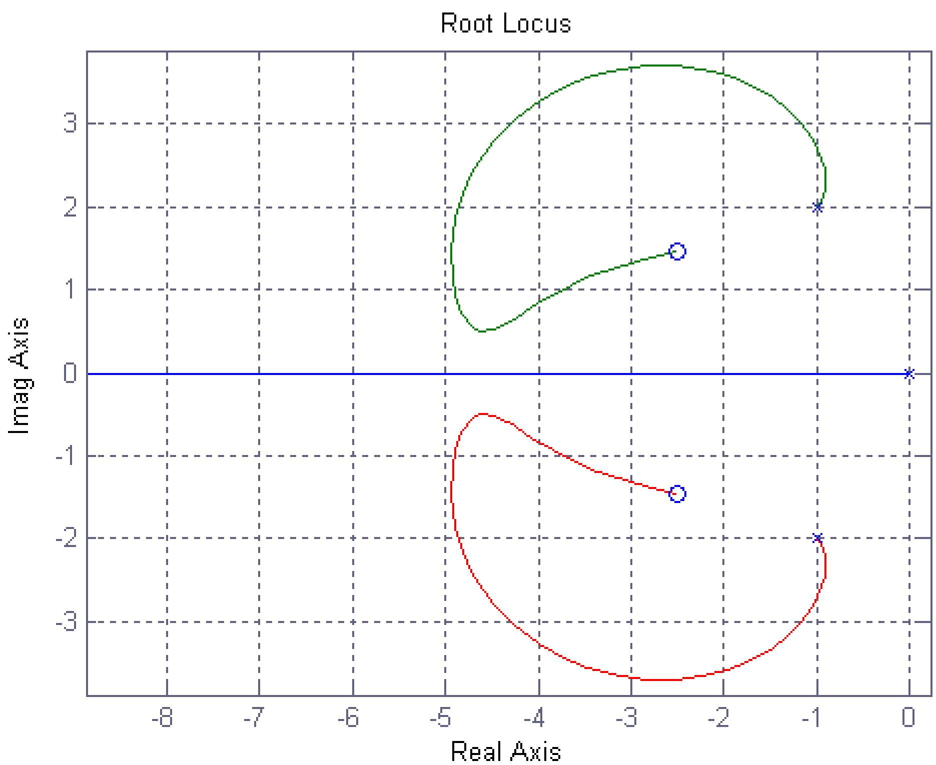


Intersection of Asymptotes:

$\sigma = \dfrac{\sum s_p - \sum s_z}{P-Z}$ Since there is only 1 asymptote: There is no intersection

Break In/Break Out Points:

$$P'Q - PQ' = 0$$
$$(3s^2 + 4s + 5)(12s^2 + 60s + 101) - (s^3 + 2s^2 + 5s)(24s + 60) = 0$$
$$12s^4 + 120s^3 + 363s^2 + 404s + 505 = 0$$
$$s_{1,2} = -4.64 \pm 0.29i$$
$$s_{3,4} = -0.36 \pm 1.35i$$



Chapter

9

CONTINUOUS CONTROL

MAE 443

Frequency Response Functions

Frequency Response Function

Frequency Response Function = TF(s = $i\omega$)   Converting Laplace to Frequency Domain

<u>Reminder:</u>

We live in the Time Domain: t

We look @ Laplace for mathematical reasons: s

Use Frequency Domain: ω

Frequency Response Functions = f(ω) by using Complex Function of ω

Frequency Response Functions = Real(ω) + Imaginary(ω)i

For given ω:

|FRF| = $\dfrac{|Output|}{|Input|}$

|FRF| = $\sqrt{\mathrm{Re}^2+\mathrm{Im}^2} = Magnitude$

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = phase$$

For a linear system, if $u = input = B\sin\varpi t$, then $y = output = A\sin(\varpi t+\phi)$

|FRF| = $\dfrac{A}{B}$ or $A = B\sqrt{\mathrm{Re}^2+\mathrm{Im}^2}$

$\therefore$ The FRF enables us to find the output for any sinusoidal input!

EXAMPLES

Example # 1 Find the Transfer Function $\frac{X(s)}{F(s)}$ for the system modeled by

$3\ddot{x} + 2\dot{x} + 4x = 5f(t)$

Answer: $\frac{X}{F} = \frac{Output}{Input} = \frac{5}{3s^2 + 2s + 4}$

Example # 2 Find the Transfer Function $\frac{Y(s)}{F(s)}$ if the output is $y(t) = 4\dot{x} + 2x$, for the

system: $\ddot{x} + 6\dot{x} + 5x = f(t) + \frac{df(t)}{dt}$

Answer: $\frac{X}{F} = \frac{s+1}{s^2 + 6s + 5}$ $\quad \therefore \quad$ $\frac{Y}{F} = \frac{(s+1)(4s+2)}{s^2 + 6s + 5} = \frac{4s^2 + 6s + 2}{s^2 + 6s + 5}$

Example # 3 Consider the Transfer Function $\frac{Y(s)}{F(s)} = \frac{s^2 + 2s}{3s^2 + 4s + 5}$

QA. Find $y(t = 0^+)$ *immediately after a step input is applied, f(t = 0) = 6, if both y = 0 and f = 0 prior to the step input.*

Answer: $6 * \lim_{s \to \infty} \frac{Y}{F} = 6 * \frac{1}{3} = 2$

QB. Find $y(t \to \infty)$ *for the same conditions in previous problem.*

Answer: $6 * \lim_{s \to 0} \frac{Y}{F} = 0$

QC. Find the Frequency Response Function of the system:

$$\text{FRF} = \frac{(i\varpi)^2 + 2(i\varpi)}{3(i\varpi)^2 + 4(i\varpi) + 5} = \frac{-\varpi^2 + 2i\varpi}{(5-3\varpi^2) + i4\varpi} * \frac{\left((5-3\varpi^2) - i4\varpi\right)}{\left((5-3\varpi^2) - i4\varpi\right)}$$

$$= \frac{3\varpi^4 + 3\varpi^2 - i(2\varpi^3 - 10\varpi)}{(5-3\varpi^2)^2 + (4\varpi)^2} = \frac{3\varpi^4 + 3\varpi^2 - i(2\varpi^3 - 10\varpi)}{9\varpi^4 - 14\varpi^2 + 25}$$

Break Up into Real and Imaginary (Not Necessary to Expand Parts)

$$\text{Real} = \frac{3\varpi^4 + 3\varpi^2}{9\varpi^4 - 14\varpi^2 + 25} \quad \text{and Imaginary } = \frac{-i(2\varpi^3 - 10\varpi)}{9\varpi^4 - 14\varpi^2 + 25}$$

QD. What are the "Zeroes" of the system:

Answer: s = 0 and s = -2

QE. What are the "Poles" of the system:

Answer: $s_{1,2} = -0.66 \pm i1.105$

QF. Find the output y(t) of the system if the input f(t) = 4sin5t ?

Answer: Substitute $\omega = 5$ into FRF

Reminder that Output = Magnitude * B = sin(5t + ϕ)

$$\text{Magnitude} = \sqrt{\text{Re}^2 + \text{Im}^2} = \sqrt{\left(\frac{3(5)^4 + 3(5)^2}{9(5)^4 - 14(5)^2 + 25}\right)^2 + \left(\frac{-2(5)^3 + 10(5)}{9(5)^4 - 14(5)^2 + 25}\right)^2} = 0.326$$

$$\phi = \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\left(\frac{-2(5)^3 + 10(5)}{3(5)^4 + 3(5)^2}\right) = 354.1 \quad 4^{\text{th}} \text{ Quadrant}$$

$\therefore$ Output = 4*0.326sin(4t + 354.1) = 1.3068 sin(4t + 354.1)

Example # 4 Consider the Transfer Function $\frac{Y(s)}{F(s)} = \frac{s+3}{s^2+2s+25}$

QA. Find $y(t=0^+)$ *immediately after a step input is applied, f(t = 0) = 1, if both y = 0 and f = 0 prior to the step input.*

Answer: $1*\lim_{s\to\infty}\frac{Y}{F} = 1*\frac{s}{s^2} = 0$

QB. Find $y(t\to\infty)$ *for the same conditions in previous problem.*

Answer: $1*\lim_{s\to 0}\frac{Y}{F} = \frac{3}{25}$

QC. Find the Frequency Response Function of the system:

$$\text{FRF} = \frac{(i\varpi)+3}{(i\varpi)^2+2(i\varpi)+25} = \frac{i\varpi+3}{(25-\varpi^2)+i2\varpi} * \frac{(25-\varpi^2)-i2\varpi}{(25-\varpi^2)-i2\varpi}$$

$$= \frac{75-\varpi^2+i(-\varpi^3+19\varpi)}{(25-\varpi^2)^2+(2\varpi)^2} = \frac{75-\varpi^2-i(-\varpi^3+19\varpi)}{\varpi^4-46\varpi^2+625}$$

Break Up into Real and Imaginary (Not Necessary to expand out parts since ω = ?)

$\text{Real} = \frac{75-\varpi^2}{\varpi^4-46\varpi^2+625}$ and Imaginary $= \frac{i(-\varpi^3+19\varpi)}{\varpi^4-46\varpi^2+625}$

QD. What are the "Zeroes" of the system:

Answer: s = - 3

QE. What are the "Poles" of the system:

Answer: $s_{1,2} = -1.0 \pm i4.89$

QF. Find the output y(t) of the system if the input f(t) = 4sin5t ?

Answer: Substitute ω = 5 into FRF

Reminder that Output = Magnitude * B = sin(5t + ϕ)

Magnitude = $\sqrt{\mathrm{Re}^2+\mathrm{Im}^2} = \sqrt{\left(\frac{75-(5)^2}{(5)^4-46(5)^2+625}\right)^2+\left(\frac{-(5)^3+19(5)}{(5)^4-46(5)^2+625}\right)^2} = 0.583$

$\phi = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{-(5)^3+19(5)}{75-(5)^2}\right) = 329.0$ 4th Quadrant

$\therefore$ Output = 4*0.583sin(4*t* + 329.0) = 2.33sin(4t + 329.0)

QG. Find the systems natural frequency?

Answer: $s^2 + 2s + 25$

$\omega_n = \sqrt{25} = 5$ rad/sec

QH. Find the systems damping ratio?

Answer: $2\xi\omega_n = 2$ $\therefore$ $\xi = 0.2$

Example # 5 Write the expression for the Transfer Function which has zeroes at (-6) and (-4), and poles at (-2 ± 4*i*)

$$TF = \frac{(s+6)(s+4)}{(s+5)(s+2+4i)(s+2-4i)} = \frac{(s+6)(s+4)}{(s+5)(s^2+4s+20)}$$

Example # 6 Write the differential model for a system whose Transfer Function $\frac{X(s)}{F(s)}$ is

$$\frac{7}{s^2+4s+3}$$

Answer: $\ddot{x}+4\dot{x}+3x = 7f(t)$

Example # 7 Write the differential model for a system whose Transfer Function $\frac{X(s)}{F(s)}$ is

$$\frac{7s}{s^2+4s+3}$$

Answer: $\ddot{x}+4\dot{x}+3x=\frac{7df(t)}{dt}$

Chapter

10

CONTINUOUS CONTROL

MAE 443

Bode and Nyquist Diagrams

Bode Plots

Transfer Function = $\dfrac{N(s)}{D(s)} = \dfrac{Polynomial\, in\, s}{Polynomial\, in\, s}\dfrac{(m^{th} - Order)}{(n^{th} - Order)}$

TF = $\left[\dfrac{(s-z_1)(s-z_2)......(s-z_m)}{(s-p_1)(s-p_2)......(s-p_n)} * \dfrac{Kn}{Kp} \right]$ $\quad z_i\text{'s} = \text{"Zeroes"}$ $\quad p_i\text{'s} = \text{"Poles"}$

Now: $20\log_{10}$ (FRF) = $20\log_{10}[N(i\omega)] - 20\log_{10}[D(i\omega)]$

$= 20\log_{10}(i\omega - z_1) + 20\log_{10}(i\omega - z_2) + \ldots\ldots .20\log_{10}(Kn)$

$- 20\log_{10}(i\omega - p_1) + 20\log_{10}(i\omega - p_2) + \ldots\ldots\ldots 20\log_{10}(Kp)$

$\therefore$ Possible to construct plot "piece-by-piece" from the numerator and denominator root terms.

Four types of terms:

1) K (constant multiplier)

2) s (root = zero)

3) s = z (or p) (real root)

4) $s^2 + 2\xi\omega_n + \omega_n^2$ (complex conjugate pairs)

Magnitude and Phase:

Type 1

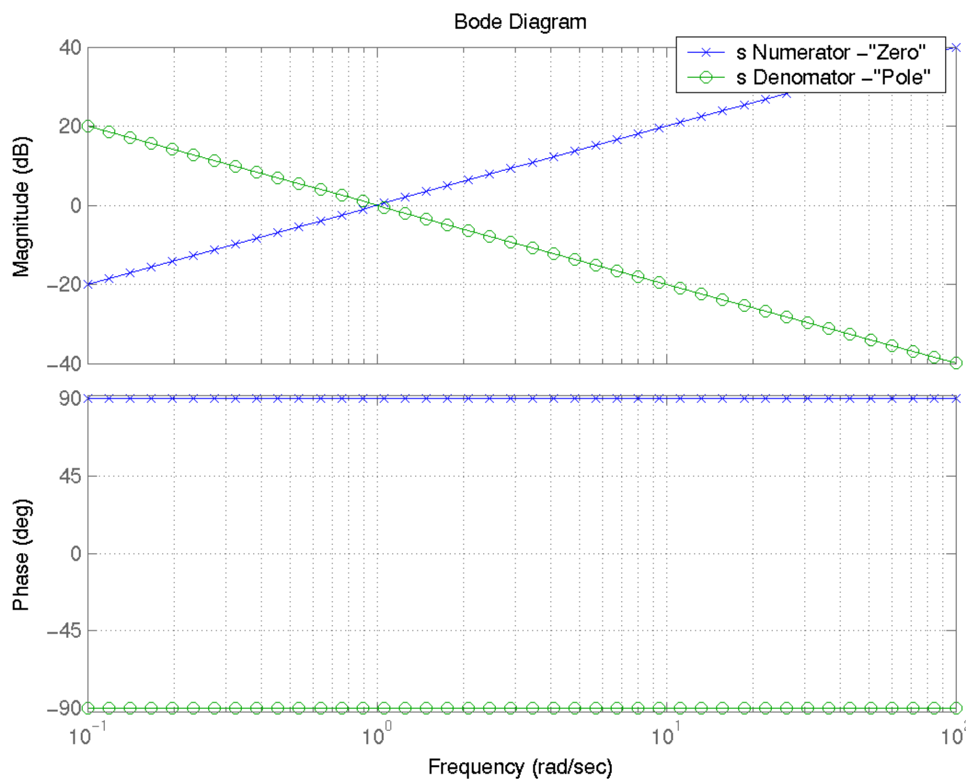
Magnitude: $20\log_{10}$ Kn = fixed numerical value for all ω

Phase: $\tan^{-1}\left(\dfrac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\dfrac{0}{K} = 0$

Type 2

Magnitude: $20\log_{10}(s = i\omega) = 20\log_{10}\omega = \begin{cases} slope = 20db \\ \varpi = 1, at 0db \end{cases}$

Phase: 2) $s = i\omega$ $\quad \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\frac{\varpi}{0} = 90$



Type 3

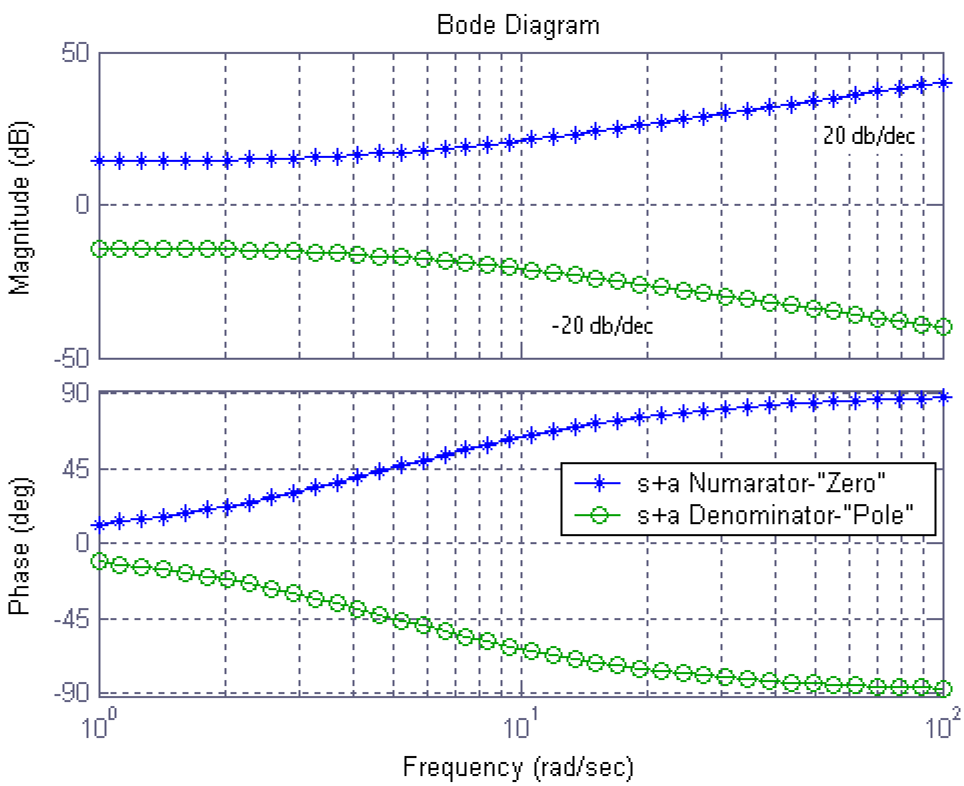
Magnitude: $20\log_{10}(i\omega + a)$, where $a = \begin{Bmatrix} -z_i \\ -p_i \end{Bmatrix}$ real root

$$= 20\log_{10}\sqrt{\varpi^2 + a^2} \approx \begin{matrix} 20\log_{10}\varpi & for\ \varpi >> a \\ 20\log_{10}a & for\ \varpi << a \end{matrix}$$

a = "Corner Frequency"   Mag = $20\log_{10} a$ $\omega << a$

Magnitude increases $20 \dfrac{db}{dec}$ $\omega >> a$

Phase: $s + a = i\omega + a$ $\tan^{-1}\left(\dfrac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\dfrac{\varpi}{a} = \begin{cases} 0 & \varpi << a \\ 90 & \varpi >> a \\ 45 & \omega = a \end{cases}$



Note: At the Corner, $20\log_{10}\sqrt{2a^2} \approx 10\log_{10} 2 + 20\log_{10} a$, where $10\log_{10} 2 = 3$ db

$\therefore$ At corner $\approx$ 3 db + $20\log_{10} a$

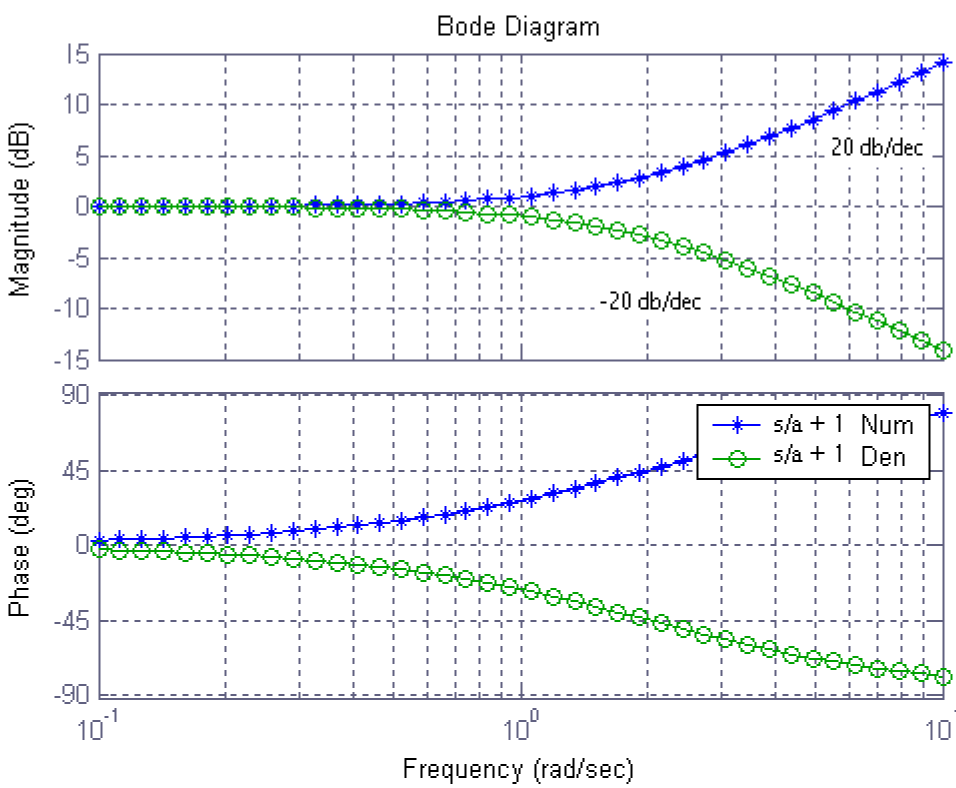
It is customary to factor out of any real root (numerator or denominator) and add the factor of K

Ex: TF = $\dfrac{(s+1)(s+5)}{(s+2)(s+3)(s+20)} = \dfrac{5\left(\frac{s}{1}+1\right)\left(\frac{s}{5}+1\right)}{120\left(\frac{s}{2}+1\right)\left(\frac{s}{3}+1\right)\left(\frac{s}{20}+1\right)}$

Now look at Magnitude in db of real root: $\left(\dfrac{s}{a}+1\right)$

$$\left(\frac{s}{a}+1\right)=\left(\frac{i\varpi}{a}+1\right)\quad\begin{cases}For \quad \varpi << a, & 20\log_{10}=0 \quad db\\ For \quad \varpi >> a, & 20\log_{10}=20 \quad db\end{cases}$$

Thus the magnitude in db = 0 below the corner frequency, makes plotting much easier.



Type 4

Magnitude: **Complex Conjugate pair of roots:** (s + a +ib)(s + a – ib)

$(s + a + ib)(s + a - ib) = (s + a)^2 + b^2 = s^2 + 2as + a^2 + b^2 = s^2 + 2\xi\omega_n + \omega_n^2$

$(s = i\omega) = -\omega^2 + i2a\omega + a^2 + b^2$

$= (a^2 + b^2) - \omega^2 + i2a\omega$

$\omega_n^2 = (a^2 + b^2)$

$$\xi = \frac{a}{\varpi_n} = \frac{a}{a^2 + b^2}$$

$\therefore \quad = \omega_n^2 - \omega^2 + i2\omega\xi\omega_n$

$$= \omega_n^2 \left[\left(1 - \frac{\varpi^2}{\varpi_n^{\,2}} \right) + i2\xi \frac{\varpi}{\varpi_n} \right]$$

$$\text{Magnitude} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \sqrt{\left(1 - \frac{\varpi^2}{\varpi_n^{\,2}} \right)^2 + \left(2\xi \frac{\varpi}{\varpi_n} \right)^2}$$

In db:

$$20\log_{10} \sqrt{\left(1 - \frac{\varpi^2}{\varpi_n^{\,2}} \right)^2 + \left(2\xi \frac{\varpi}{\varpi_n} \right)^2}$$

for $\omega \ll \omega_n$: Magnitude = 1, $\therefore$ db = 0

for $\omega \gg \omega_n$: Magnitude = $\sqrt{\left(\frac{\varpi}{\varpi_n} \right)^4} = \left(\frac{\varpi}{\varpi_n} \right)^2$

in db, $20\log_{10} \left(\frac{\varpi}{\varpi_n} \right)^2 = 40\log_{10}\left(\frac{\varpi}{\varpi_n} \right) = 40 \frac{db}{dec}$ slope

ω_n : is the corner frequency

Note: Unlike K, s, or s + a terms, the complex terms can increase (or decrease) in magnitude at the corner frequency, without limit!

At $\omega = \omega_n$, Re = $\left(1 - \frac{\varpi^2}{\varpi_n{}^2}\right) = 0$ and Im = $2\xi \frac{\omega}{\varpi_n} = 2\xi$

$\therefore$ Magnitude at $\omega = \omega_n$ in db

$$20\log_{10} \sqrt{(2\xi)^2} = 20\log_{10} 2\xi = \begin{cases} > 0 & if\,(2\xi) > 1 \\ = 0 & if\,(2\xi) = 1 \\ < 0 & if\,(2\xi) < 1 \end{cases}$$

An Underdamped System

If the transfer function given by $T(s) = \dfrac{1}{\frac{m}{k}s^2 + \frac{c}{k}s + 1} = \dfrac{1}{\left(\frac{2}{\omega_n}\right)^2 + 2\xi\frac{s}{\omega_n} + 1}$

By replacing s with iω, $T(i\varpi) = \dfrac{1}{\left(\frac{i\varpi}{\varpi_n}\right)^2 + \frac{2\xi}{\omega_n}\varpi i + 1} = \dfrac{1}{1 - \left(\frac{\varpi}{\varpi_n}\right)^2 + \frac{2\xi\varpi}{\omega_n} i}$

The magnitude : $M(\varpi) = \dfrac{1}{\sqrt{\left(1 - \frac{\varpi^2}{\varpi_n{}^2}\right)^2 + \left(\frac{2\xi\varpi}{\omega_n}\right)^2}}$

This has a maximum value when the denominator has a minimum. Setting the derivative of the denominator with respect to ω equal to zero shows that the maximum M(ω) occurs at $\omega = \omega_n \sqrt{1 - 2\xi^2}$. This frequency is the resonant frequency ω_r. The peak of M(ω) exist only when the term under the radical is positive; that is when $\xi \leq 0.707$, thus

$$\omega_r = \varpi_n \sqrt{1 - 2\xi^2} \qquad 0 \leq \xi \leq 0.707$$

The value of the peak M_p is found by substituting ω_r int M(ω). This gives:

$$M_p = M(\varpi_r) = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad 0 \le \xi \le 0.707$$

If $\xi > 0.707$, no peak exists, and the maximum value of M occurs at $\omega = 0$ where $M = 1$.

Note that as $\xi \to 0, \varpi_r \to \varpi_n and M_p \to \infty$. For an undamped system, the resonant frequency is the natural frequency ω_{n}.

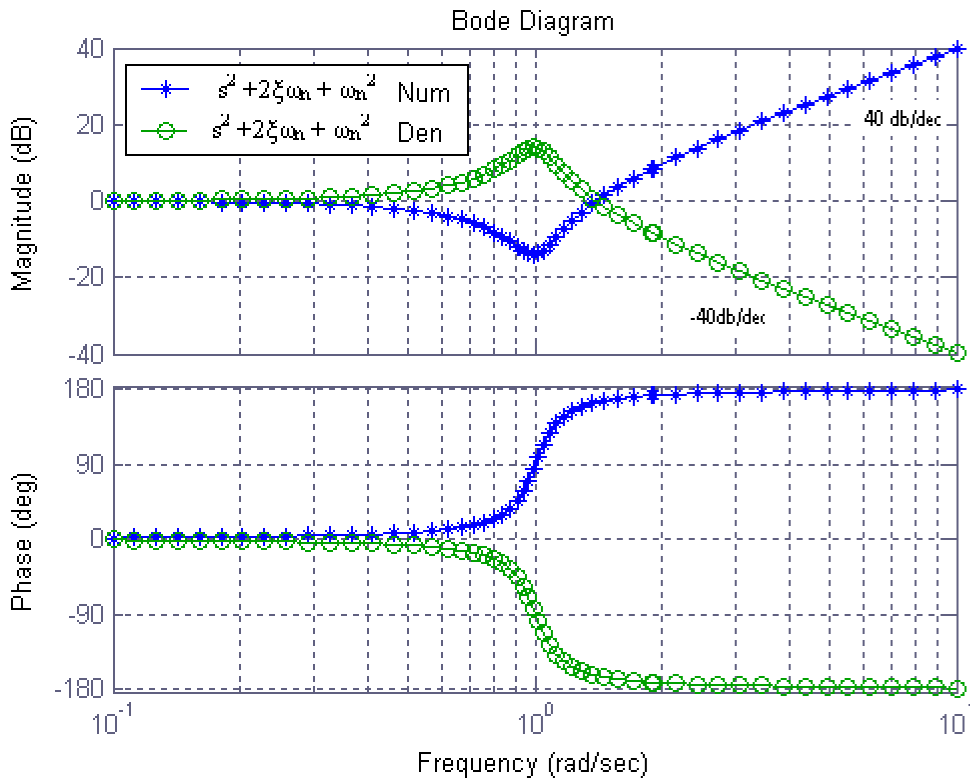
Phase: $s^2 + 2\xi\omega_n + \omega_n^2 = \omega_n^2 \left[\left(1 - \frac{\varpi^2}{\varpi_n^2}\right) + i2\xi\frac{\varpi}{\varpi_n}\right]$

$$\tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\left(\frac{2\xi\frac{\varpi}{\omega_n}}{1-\frac{\varpi^2}{\varpi_n^2}}\right)$$

If $\omega \ll \omega_n$, $\tan^{-1}\left(\frac{0}{1}\right) = 0$

If $\omega \gg \omega_n$, $\tan^{-1}\left(\frac{0}{-\infty}\right) = 180$

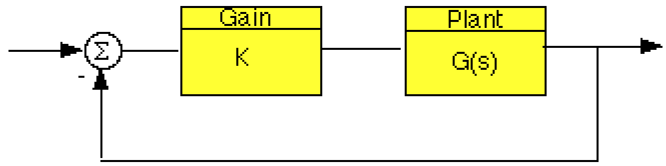
If $\omega = \omega_n$, $\tan^{-1}\left(\frac{2\xi}{0}\right) = 90$



INSERT DIFFERENT TYPES OF ZETA

Gain and Phase Margin

Let's say that we have the following system:



Where K is a variable (constant) gain and G(s) is the plant under consideration. The **<u>gain margin</u>** is defined as the change in open loop gain required to make the system unstable.

Systems with greater gain margins can withstand greater changes in system parameters before becoming unstable in closed loop.

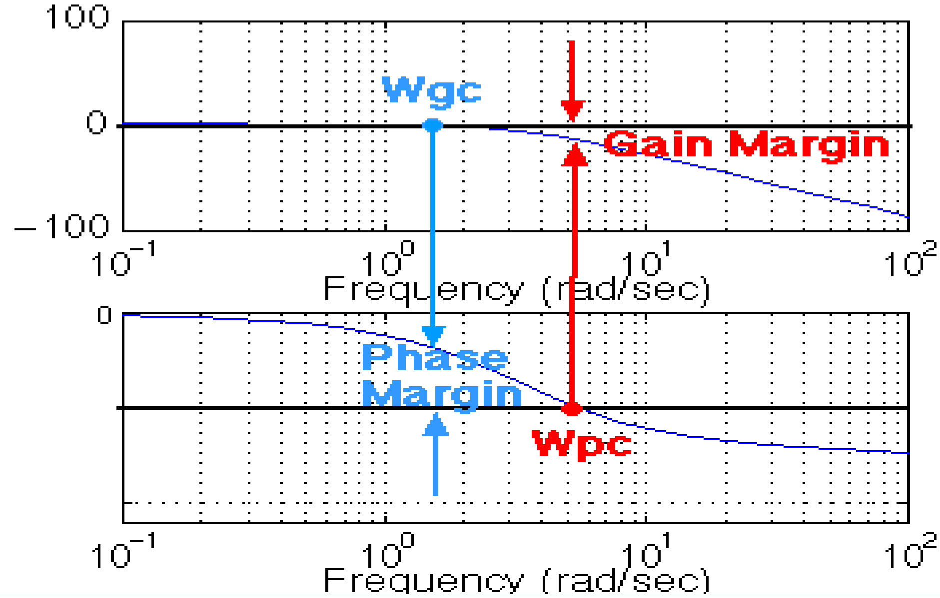
Keep in mind that unity gain in magnitude is equal to a gain of zero in dB.

The **phase margin** is defined as the change in open loop phase shift required to make a closed loop system unstable.

The phase margin also measures the system's tolerance to time delay. If there is a time delay greater than 180/Wpc in the loop (where **Wpc** is the frequency where the phase shift is 180 deg), the system will become unstable in closed loop. The time delay can be thought of as an extra block in the forward path of the block diagram that adds phase to the system but has no effect the gain. That is, a time delay can be represented as a block with magnitude of 1 and phase w*time_delay (in radians/second).

For now, we won't worry about where all this comes from and will concentrate on identifying the gain and phase margins on a Bode plot:

The phase margin is the difference in phase between the phase curve and -180 deg at the point corresponding to the frequency that gives us a gain of 0dB (the gain cross over frequency, **Wgc**). Likewise, the gain margin is the difference between the magnitude curve and 0dB at the point corresponding to the frequency that gives us a phase of -180 deg (the phase cross over frequency, **Wpc**).

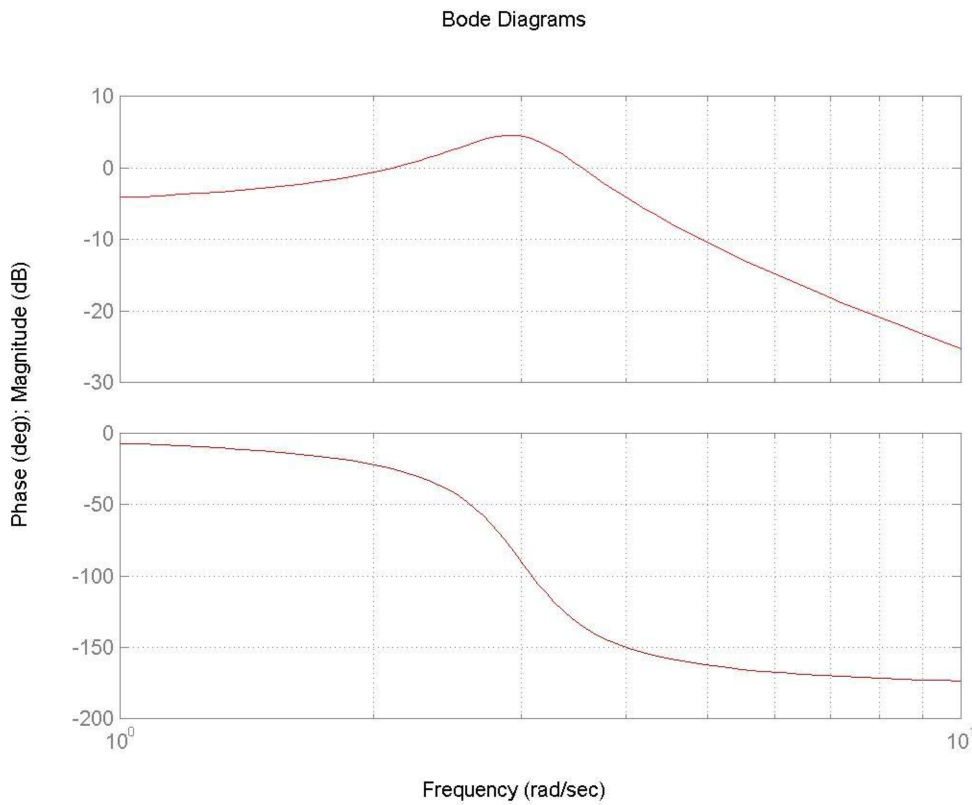


INSERT NYQUIST PLOT to show wpc,phase margin,etc…..

EXAMPLES

Example # 1 Answer the following questions for the Bode plots

QA. Estimate the system transfer function



You know that the denominator's order is 2 more that numerator as it ends at -180. We see no rise of the phase plot therefore it has "no zeroes" and using the magnitude plot we see that $\xi \approx 0.2$ and that $\omega_n \approx 3$ rad/sec

$\therefore\ TF = \dfrac{Kn}{s^2 + 2\xi\varpi_n s + \varpi_n^{\ 2}}$ and at 1 db ≈ - 4 db $\therefore$ $20\log_{10}(k) = -4$ db , k = .604

$k = \dfrac{Kn}{\varpi_n^{\ 2}} \qquad \therefore \qquad Kn = k\varpi_n^{\ 2} = 0.604*9 = 5.67$

$\therefore \quad TF = \dfrac{5.67}{s^2 + 1.25s + 9}$

*QB. What is the output if the input is u = 5 * sin (4t)?*

Input is k*magnitude*sin(bt + ϕ)

Look on the plot at 4 rad/se and read the magnitude: magnitude = - 4 db

$\therefore$ $20\log_{10}(k)$ = - 4 db , k = .604 and ϕ at 4 rad/sec ≈ -150

$\therefore$ Output = 5(.604)sin(4t – 150)

Output = 3.15sin(4t – 150)

QC. At what frequency(ies) (if any) is the output magnitude equal to the input magnitude?

Look on the magnitude plot, find where the plot crosses 0 db

$\therefore$ ≈ 2 and 3.5 rad/sec

*QD. Assume that the input to the system is u = b * sin (w*t), and that b is a constant while w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output the greatest?*

Find on the magnitude plot the greatest height = 3 rad/sec

QE. At what frequency (if any) is the output magnitude exactly one-tenth the size of the greatest output magnitude?

1/10 of the greatest magnitude is Highest db – 20 db

1/100 of the greatest magnitude is Highest db – 40 db

$\therefore$ Highest is 5 db – 20 db = -15 db which is at 6.0 rad/sec

QF. At what frequency if any is the output exactly out of phase with the input?

Look at phase plot and see where it touches -180 $\therefore \omega \geq 10$ rad/sec

Example # 2 $TF = \dfrac{s+2}{s^2+3s+25}$

Break up into $TF = \dfrac{2\left(\dfrac{s}{2}+1\right)}{25\left(\dfrac{s^2}{25}+\dfrac{3s}{25}+1\right)}$

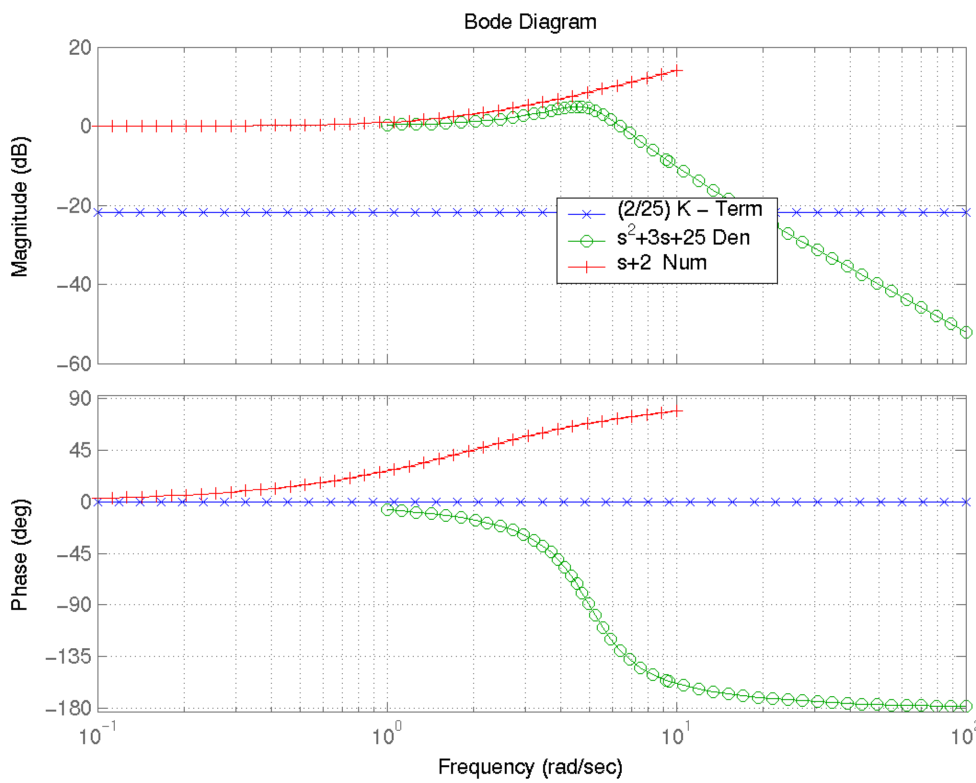
Find K term $\therefore$ $20\log_{10}(2/25) = -21.9$ db

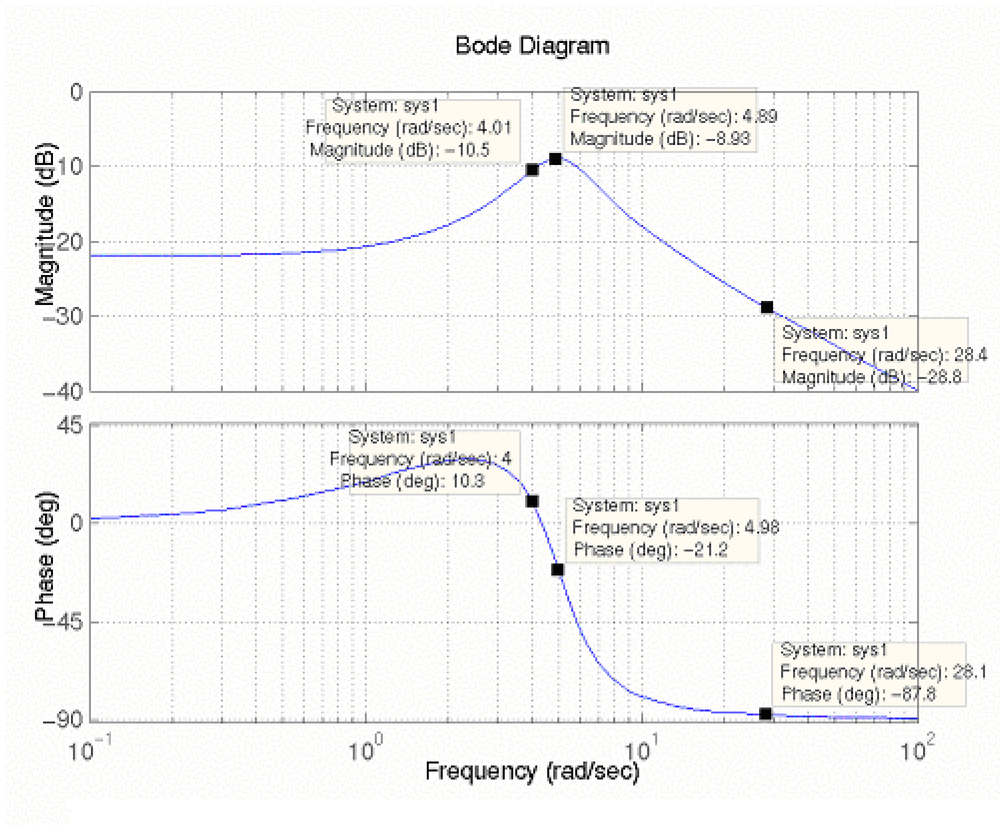
Numerator term of (s + 2)

Denominator Term $s^2 + 3s + 25$

$\omega_n{}^2 = 25 \quad \therefore \omega_n = 5$ and $2\xi\omega_n = 3 \;\; \therefore \;\; \xi = .3$

Look on magnitude plot for $\xi = .3$ and find how high the magnitude rise and place that hump at 5 rad/sec. Now it's time to plot the parts and combine.





*QB. What is the output if the input is u = 5 * sin (4t)?*

Input is k*magnitude*sin(bt + ϕ)

Look on the plot at 4 rad/se and read the magnitude: magnitude = - 10 db

$\therefore$ $20\log_{10}(k) = -10$ db , k = .316 and ϕ at 4 rad/sec $\approx$ + 10

$\therefore$ Output = 5(.316)sin(4t + 10)

Output = 1.59sin(4t + 10)

QC. At what frequency(ies) (if any) is the output magnitude equal to the input magnitude?

Look on the magnitude plot, find where the plot crosses 0 db

$\therefore$ None for this particular TF

*QD. Assume that the input to the system is u = b * sin (w*t), and that b is a constant while w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output the greatest?*

Find on the magnitude plot the greatest height = 5 rad/sec

QE. At what frequency (if any) is the output magnitude exactly one-tenth the size of the greatest output magnitude?

1/10 of the greatest magnitude is Highest db – 20 db

1/100 of the greatest magnitude is Highest db – 40 db

$\therefore$ Highest is -9 db – 20 db = -29 db which is at 30.1 rad/sec

QF. At what frequency if any is the output exactly out of phase with the input?

Look at phase plot and see where it touches -180 $\therefore$ None for this TF

Example # 3 $TF = \dfrac{s^2 + 2s + 64}{s^2 + 30s + 6400}$

Break up into $TF = \dfrac{64\left(\dfrac{s^2}{64} + \dfrac{2s}{64} + 1\right)}{6400\left(\dfrac{s^2}{6400} + \dfrac{30s}{6400} + 1\right)}$

Find K term $\therefore$ $20\log_{10}(64/6400)$ = - 40 db

Numerator term of $s^2 + 2s + 64$

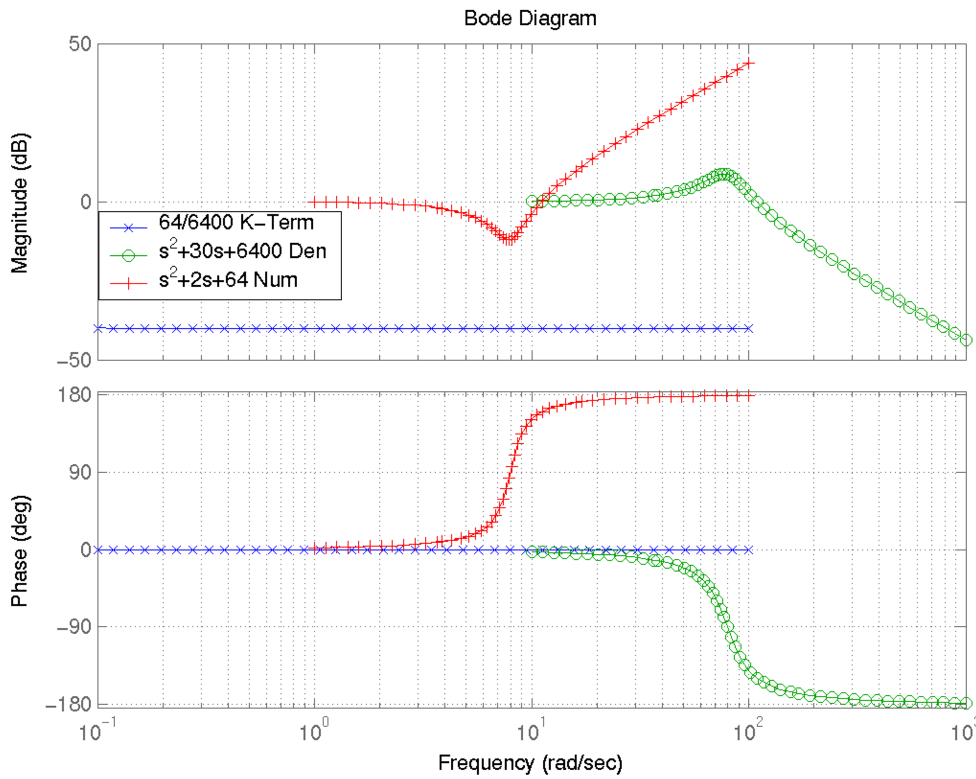
$\omega_n^{\ 2} = 64 \quad \therefore\ \omega_n = 8$ and $2\xi\omega_n = 2 \quad \therefore\ \xi = .125$

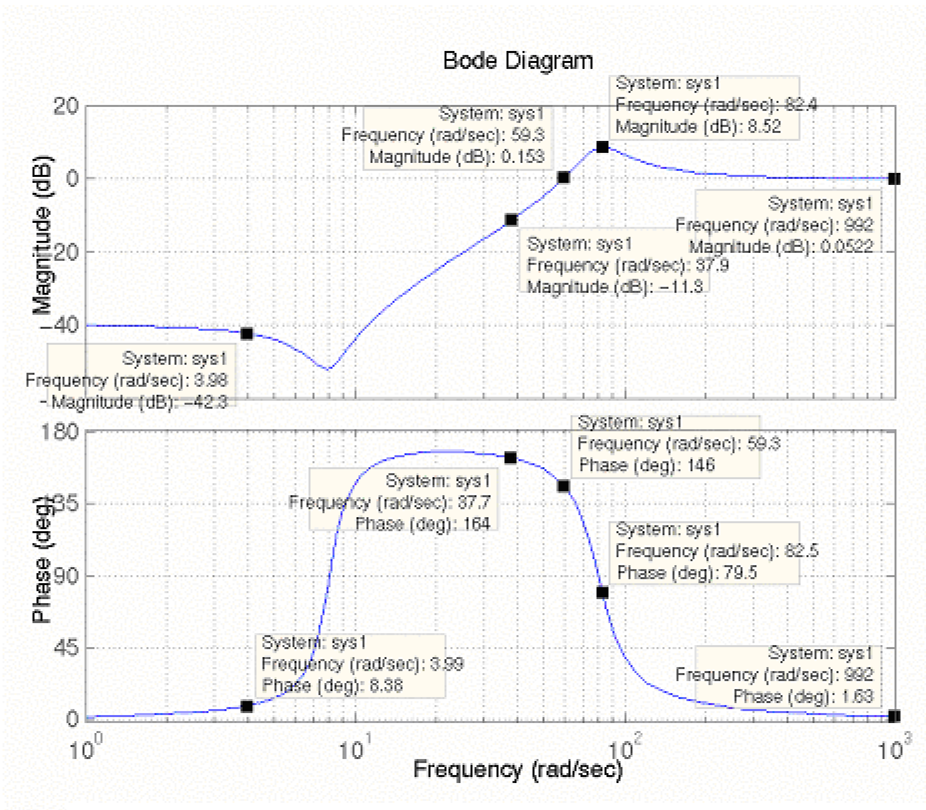
Look on magnitude plot for $\xi = .125$ and find how high the magnitude rise and place that hump at 8 rad/sec. <u>This must be drawn upside down since it's in numerator!!!!</u>

Denominator Term $s^2 + 30s + 6400$

$\omega_n{}^2 = 6400 \quad \therefore \ \omega_n = 80$ and $2\xi\omega_n = 30 \ \therefore \ \xi = .188$

Look on magnitude plot for $\xi = .188$ and find how high the magnitude rise and place that hemp at 80 rad/sec. Now it's time to plot the parts and combine.





*QB. What is the output if the input is u = 5 * sin (4t)?*

Input is k*magnitude*sin(bt + ϕ)

Look on the plot at 4 rad/se and read the magnitude: magnitude = - 42 db

$\therefore$ $20\log_{10}(k) = -42$ db , k = .007 and ϕ at 4 rad/sec $\approx +8$

$\therefore$ Output = 5(.007)sin(4t + 8)

Output = 0.035(4t + 8)

QC. At what frequency(ies) (if any) is the output magnitude equal to the input magnitude?

Look on the magnitude plot, find where the plot crosses 0 db

$\therefore$ ≈ 60 and $\geq 1{,}000$ rad/sec

*QD. Assume that the input to the system is u = b * sin (w*t), and that b is a constant while w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output the greatest?*

Find on the magnitude plot the greatest height = 83 rad/sec

QE. At what frequency (if any) is the output magnitude exactly one-tenth the size of the greatest output magnitude?

1/10 of the greatest magnitude is Highest db – 20 db

1/100 of the greatest magnitude is Highest db – 40 db

$\therefore$ Highest is 9 db – 20 db = - 11 db which is at 36.1 rad/sec

QF. At what frequency if any is the output exactly out of phase with the input?

Look at phase plot and see where it touches -180 $\therefore$ None for this TF

Example # 4 $TF = \dfrac{s(s+1)}{(s+10)(s^2+30s+6400)}$

Break up into $TF = \dfrac{1s\left(\dfrac{s}{1}+1\right)}{10*6400\left(\dfrac{s}{10}+1\right)\left(\dfrac{s^2}{6400}+\dfrac{30s}{6400}+1\right)}$

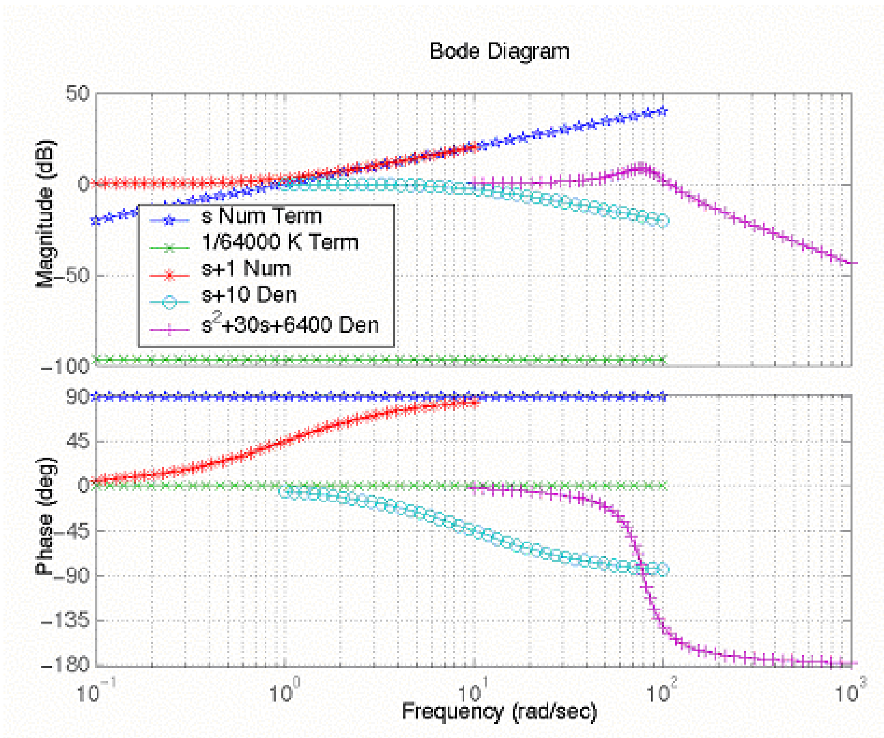
Find K term $\therefore$ $20\log_{10}(1/64000)$ = - 96.1 db

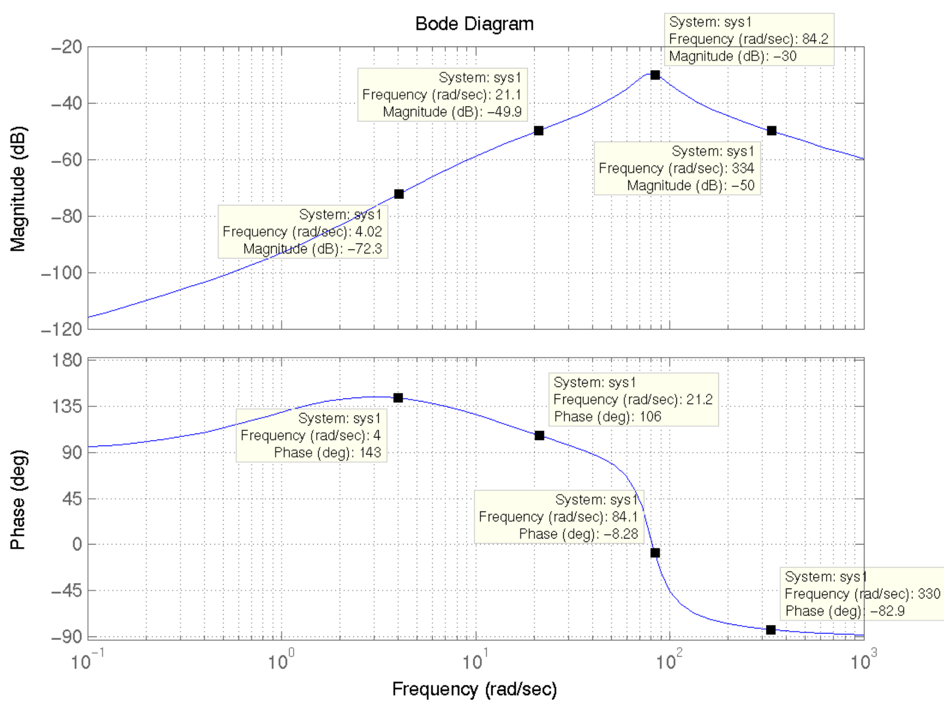
Numerator term of has (s + 1) and basic s term

Denominator Term (s + 10) and $s^2 + 30s + 6400$

$\omega_n{}^2 = 6400 \quad \therefore \omega_n = 80$ and $2\xi\omega_n = 30 \ \therefore \ \xi = .188$

Look on magnitude plot for $\xi = .188$ and find how high the magnitude rise and place that hump at 80 rad/sec. Now it's time to plot the parts and combine.





QB. What is the output if the input is $u = 5 * \sin(4t)$?

Input is k*magnitude*sin(bt + ϕ)

Look on the plot at 4 rad/se and read the magnitude: magnitude = - 72 db

$\therefore$ $20\log_{10}(k) = -72$ db , k = .00025 and ϕ at 4 rad/sec ≈ + 145

$\therefore$ Output = 5(.00025)sin(4t + 145)

Output = 0.012(4t + 145)

QC. At what frequency(ies) (if any) is the output magnitude equal to the input magnitude?

Look on the magnitude plot, find where the plot crosses 0 db

$\therefore$ None cross 0 db for this particular TF

QD. Assume that the input to the system is $u = b * \sin(w*t)$, and that b is a constant while w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output the greatest?

Find on the magnitude plot the greatest height = 84 rad/sec

QE. At what frequency (if any) is the output magnitude exactly one-tenth the size of the greatest output magnitude?

1/10 of the greatest magnitude is Highest db – 20 db

1/100 of the greatest magnitude is Highest db – 40 db

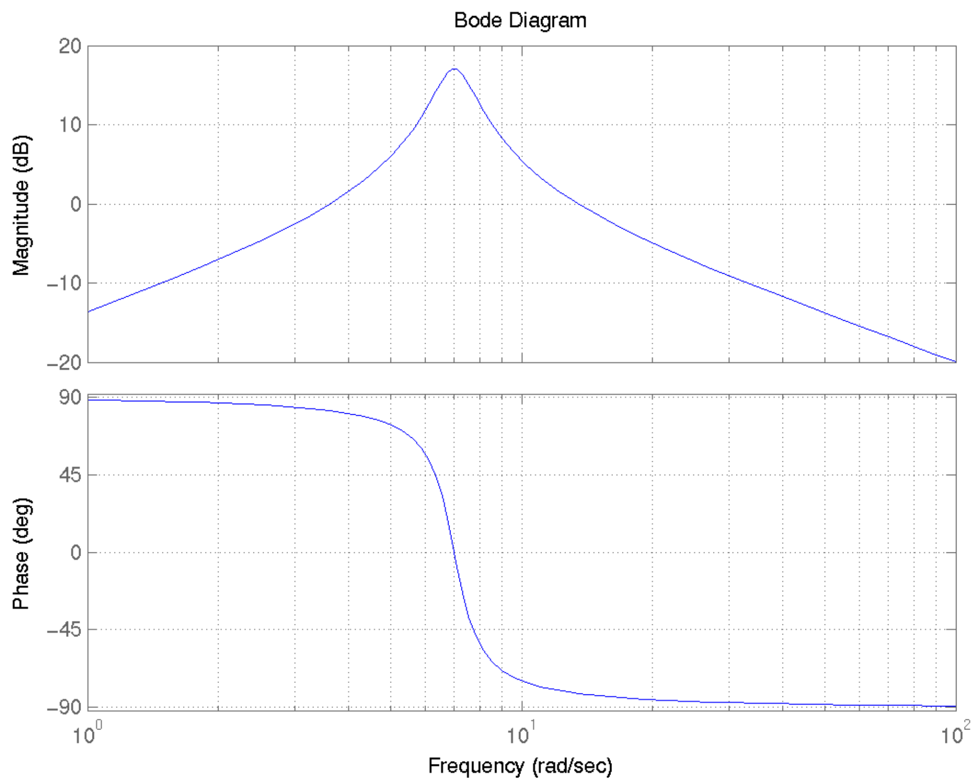
$\therefore$ Highest is -29 db – 20 db = - 49 db which is at 21.1 and 333 rad/sec

QF. At what frequency if any is the output exactly out of phase with the input?

Look at phase plot and see where it touches -180 $\therefore$ None for this TF

Ex #5 Answer the following questions for the Bode plots

QA. Estimate the system transfer function



We know that the phase starts off at 90 so you know that there is a "s" term in numerator. You also know that the denominator is second order as it goes from +90 to -90 on phase plot and also see on the magnitude plot a decreasing line of 40 db/dec. We can also conclude that $\xi \approx 0.1$ and that $\omega_n \approx 7$ rad/sec so the TF is going to look like:

$\therefore\ TF = \dfrac{Kns}{s^2 + 2\xi\varpi_n s + \varpi_n^{\,2}}$ and at 1 db ≈ - 13.1 db $\therefore$ $20\log_{10}(k) = -13.1$ db , k = 0.22

$$k = \frac{Kn}{\varpi_n^{\,2}} \qquad \therefore \qquad Kn = k\varpi_n^{\,2} = 0.22 * 49 = 10.78$$

$$\therefore \quad TF = \frac{10.78s}{s^2 + 1.4s + 49}$$

*QB. What is the output if the input is u = 3 * sin (6t)?*

Input is k*magnitude*sin(bt + ϕ)

Look on the plot at 6 rad/se and read the magnitude: magnitude = 11.8 db

$\therefore$ $20\log_{10}(k) = 11.8$ db , k = 3.98 and ϕ at 6 rad/sec ≈ + 56.5

$\therefore$ Output = 3(3.98)sin(6t + 56.5)

Output = 11.67sin(6t + 56.5)

*QC. What input will produce an output of y = 10 * sin (10t)?*

Now we must work backwards to find the input:

Look on plots at ω ≈ 10 rad/sec which gives 5.5 db and ϕ = -85

$\therefore$ $20\log_{10}(k) = 5.5$ db , k = 1.89

In General: input = Rsin(bt + ϕ)

$\therefore$ 1.89*B = R = 10 $\therefore$ B = 5.3 and we also know that input ϕ = 0

$\therefore$ Input is: 5.3sin(10t + 85)

QD. At what frequency(ies) (if any) is the output magnitude equal to the input magnitude?

Look on the magnitude plot, find where the plot crosses 0 db

$\therefore$ ≈ 3.6 and 13.6 rad/sec

*QE. Assume that the input to the system is u = 7 * sin (w*t), and w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output the greatest?*

Find on the magnitude plot the greatest height = 7 rad/sec

*QF. Assume that the input to the system is u = 7 * sin (w*t), and w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output equal to 15?*

$\therefore$ $20\log_{10}(15/7) = 6.62$ db $\therefore$ look at plot and find $\omega \approx 5.1$ and 9.6 rad/sec

*QG. Assume that the input to the system is u = 24 * sin (w*t), and w can vary over the range shown in the Bode plot. At what frequency (if any) is the output magnitude exactly one-tenth the size of the greatest output magnitude?*

1/10 of the greatest magnitude is Highest db – 20 db

1/100 of the greatest magnitude is Highest db – 40 db

$\therefore$ Highest is 17 db – 20 db = -3 db which is at 2.9 and 17 rad/sec

QH. At what frequency if any is the output exactly out of phase with the input?

Look at phase plot and see where it touches -180 $\therefore$ No frequencies for this TF

QI. At what frequency if any is the output exactly in phase with the input?

Look on the plot where you see zero degree difference $\therefore$ $\omega \approx 7$ rad/sec

QI. What is the approximate time constant of the system?

$$\therefore \tau = \frac{1}{\xi \varpi_n} = \frac{1}{0.1*7} = 1.43 \text{ seconds}$$

Example # 6 Plot by hand the Bode magnitude and phase angle plot for the system with zero at s = 1 and poles at s = - 1 , - 2 ± i*10, all multiplied by a constant of 1000.

$$TF = 1000*\left(\frac{(s-1)}{(s+1)(s+2+i10)(s+2-i10)}\right) = \frac{1000(s-1)}{(s+1)(s^2+4s+104)}$$

Break up into $TF = \dfrac{1000\left(\dfrac{s}{1}-1\right)}{104\left(\dfrac{s}{1}+1\right)\left(\dfrac{s^2}{104}+\dfrac{4s}{104}+1\right)}$

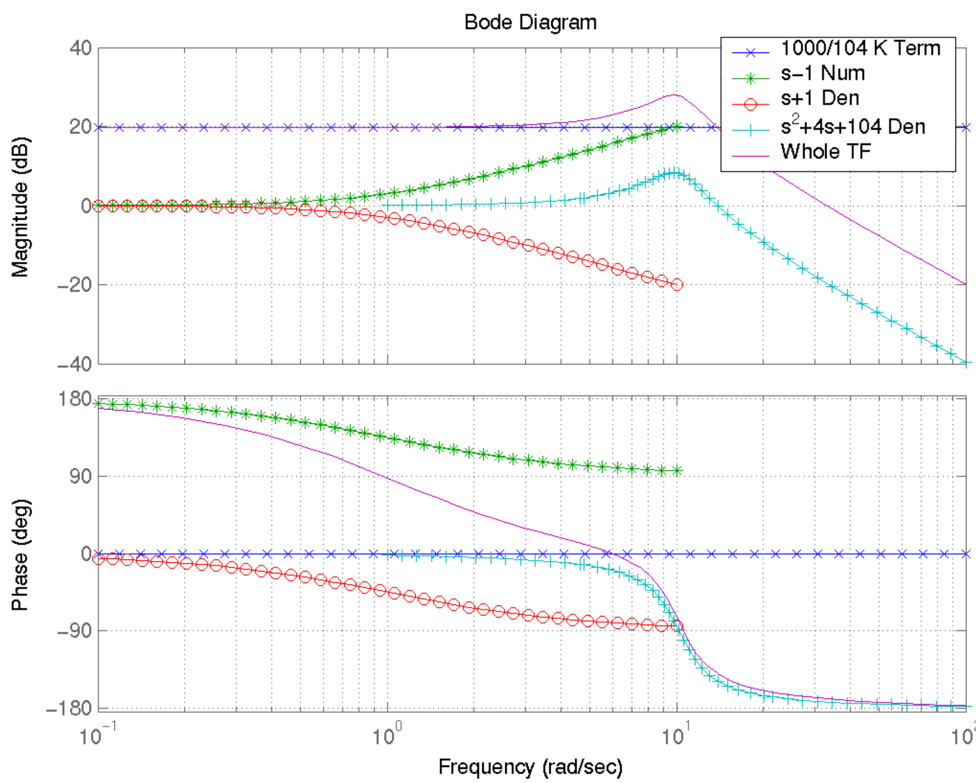
Find K term $\therefore$ $20\log_{10}(1000/104) = 9.615$ db

Numerator term of has (s - 1); be very careful on phase with a minus term

Denominator Term (s + 1) and $s^2 + 4s + 104$

$\omega_n{}^2 = 104 \quad \therefore \omega_n = 10.2$ and $2\xi\omega_n = 4 \quad \therefore \quad \xi = 0.2$

Look on magnitude plot for $\xi = 0.2$ and find how high the magnitude rise and place that hump at 10.2 rad/sec. Now it's time to plot the parts and combine



CT(sys), where sys is a LTI SISO system transfer function. Transfer functions are expressed in the form they are commonly written in the Laplace domain that is as function of s, for example (s^2+s+1)/((s+3)(s+5)). CT supports DELAYS!!! For SISO systems multiply the system for e^(-delay*s). For example e^(-0.03s)*(s+1)/(s^2*(s^2+s+100)).
Menu explanation:

- BODE TABLE memorizes the amplitude (in db) and the phase (in degree) functions in y1(x) and y2(x), then display a table of logarithmically or linear equally spaced points within the selected frequencies; by default is selected 10 points for each decade of the calculated range of the plot.
- BODE DIAGRAM displays the bode plot of the amplitude or of the phase. 'Grid' is the logarithmic grid, 'Res' is the resolution of the plot. A proper range for the plot is automatically calculated.
- NICHOLS DIAGRAM displays the Nichols plot of the system. 'Res' is the resolution of the plot. The x-axis is the amplitude in db, the y-axis is the phase in degree. You can also diagram the constant amplitude closed loop line writing the number in db in the 'closed loop' dialog. For example write '-3' to diagram the line at -3db.
- NYQUIST DIAGRAM displays the Nyquist plot of the system. Check 'Symmetric plot' if you want to display also the symmetric curve. 'Res' is the resolution of the plot. The x-axis is the real part of the transfer function valuated in s=i*omega, the y-axis is the imaginary part.
- MARGIN returns the gain margin, the phase margin, and associated frequencies Wcg and Wcp, given the LTI SISO open-loop model SYS. The gain margin is defined as 1/G where G is the gain at the -180 phase frequency. The gain margin is in dB, the phase margin is in degree. Note: if there is more than one crossover point, margin will return the worst case gain and phase margins.
- ABOUT displays an about dialog!
- SAVE&EXIT exits without deleting all the variable used: y1, y2, xt1, yt1, xt2, yt2, xt3, yt3, tblInput and restoring the previous settings.

After the selected diagram it's plotted by the toolbar at the top you can:
- return to the main menu pressing ESC;
- save the plot as a PIC file (press <F1>+<1>);
- change the zoom (press <F2>);
- get a point info (press <F3>);
To exit from the program restoring the previous settings and functions press ESC.
CT need the private functions MAG1 and PHASE1, LOGSPACE and ZOOMFIT2 .

RLOCUS(SYS,kLIST), where SYS is a LTI SISO system transfer function, kLIST the list of gains. It plots the locus of the roots of:
H(s) = D(s) + k * N(s) = 0,
where N(s) and D(s) are the numerator and denominator of SYS, for the list of gains. It also displays the PZMAP (see).
TIP: you can use the built-in function seq(instead of writing manually the list kLIST.
RLOCUS need ZPKDATA, GETTD, ROOTS, CPOLES and POLY2COF
Usage example
rlocus((s+1)/(s+10),{1,2,3,4,5})

ROUTH(POLY,VAR), where POLY is a polynomial in the variable VAR. Returns the Routh-Hurwitz matrix. During the compilation of the rows of the matrix if there is a zero only in the first column it is replaced with the constant epsilon. Instead if there is a row of zeroes first it forms an expression based on the row immediately above the row of zeroes, applying the coefficients to their matching powers of s, then it takes the derivative of this expression and it uses the coefficients of the new expression in the row of zeroes.
ROUTH need the function POLY2COF.
Usage example
routh(s^4+s^3+s^2+s+1,s) returns
[[1,1,1][1,1,0][eps,1,0][(eps-1)/eps,0,0][1,0,0]]

SS(A,B,C,D), where A,B,C,D are the matrices of the system:
x' = Ax + Bu
y = Cx + Du
Returns the LTI SISO transfer function:
W(s) = NUM(s) / DEN(s) = C * (sI - A)$^{-1}$ * B + D
You can also enter an expression instead of a matrix if it contains only one element and you can enter '0' instead of the matrix D if it is empty.
Usage example
ss([1,2;3,4],[1;2],[0,1],0) returns
(2*s+1)/(s^2-5*s-2)

TF2SS(SYS), where SYS is a LTI SISO system transfer function. Displays the matrixes A,B,C,D of the system:
W(s) = NUM(s) / DEN(s) = C * (s*I - A)$^{-1}$ * B + D
It also prompts to save the matrices in the variables a,b,c,d in the current directory; note that tf2ss isn't a function, it's a program!
TF2SS need the functions TF2SS2, POLY2COF.
Usage example
tf2ss((2*s+1)/(s^2-5*s-2)) displays and creates the matrices
a=[[0,1][2,5]]
b=[[0][1]]
c=[[1,2]]
d=0

TF2SS2(SYS), where SYS is a LTI SISO system transfer function. Returns a list of 4 elements with the strings of the A,B,C,D matrices of the system. TF2SS2 need the function POLY2COF.
Usage example
tf2ss2((2*s+1)/(s^2-5*s-2)) returns
{"[[0,1][2,5]]","[[0][1]]","[[1,2]]","0"}